International Journal of Pure and Applied Mathematics
Volume 16 No. 4 2004, 533-535

A VANISHING THEOREM FOR THE COHOMOLOGY OF POWERS OF IDEAL SHEAVES OF SPACE CURVES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we show the existence of many pairs (d, g) such that there is a smooth curve $C \subset \mathbf{P}^3$ with degree d and genus g for which we may give a reasonable upper bound for the index of regularity of every power $(\mathcal{I}_C)^k$, $k > 0$. All our curves are contained in a smooth quartic surface.

AMS Subject Classification: 14F17, 14H60, 14N05
Key Words: infinitesimal neighborhoods, regularity of powers of ideal sheaves, ideal sheaves, vanishing theorems, space curves, $K3$ surface, index of regularity

1. Powers of the Ideal Sheaves of Space Curves

Let $C \subset \mathbf{P}^n$ be a smooth curve and $k > 0$ an integer. The infinitesimal neighborhood $C^{(k-1)}$ of order $k-1$ of C in $\mathbf{P}^n$ is the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_C)^k$ as its ideal sheaf. The cohomology groups $h^i(\mathbf{P}^n, \mathcal{I}_{C^{(k-1)}}(t))$, $i \geq 0$,

Received: September 11, 2004

$k > 0$, $t \in \mathbb{Z}$ are important projective invariants of C. Many papers were devoted to the proof of sharp vanishing theorems of $h^1(\mathbf{P}^n, \mathcal{I}_{C^{(k-1)}}(t))$ when C is a linearly normal curve with $\deg(C) \gg p_a(C) + k$ (see [7], [6] and references therein). Here we will consider the case $n = 3$ and $p_a(C) > \deg(C) - 3$ and prove the following result.

Theorem 1. *Fix integers k, d, g such that $k > 0$, $d \geq 5$ and $d - 4 < g < (d^2 - 4)/8$. Set $r := [(d - \sqrt{d^2 - 8g})/4]$, $d_0 := d - 4r$, $g_0 := 2r^2 - dr + g$ and $t(d, g, k) := \max\{k(r+d_0), 4k+d-7\}$. Then there exists a smooth and connected curve $C \subset \mathbf{P}^3$ such that $\deg(C) = d$, $p_a(C) = g$ and $h^1(\mathbf{P}^3, (\mathcal{I}_C)^k)(t)) = 0$ for every $t \geq t(d, g, k)$.*

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Remark 1. Let $S \subset \mathbf{P}^3$ an integral degree x surface and $C \subset S$ a smooth curve such that $C \cap \mathrm{Sing}(S) = \emptyset$. For any closed subscheme Z of $\mathbf{P}^3$, let $\mathrm{Res}_S(Z)$ denotes the residual scheme of Z with respect to S, i.e. the closed subscheme of $\mathbf{P}^3$ with $\mathcal{I}_Z : \mathcal{I}_S \cong \mathcal{I}_Z : \mathcal{O}_{\mathbf{P}^3}(-x)$ as its ideal sheaf. By the very definition of residual scheme, for every integer t there is an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_S(Z)}(t - x) \to \mathcal{I}_Z(t) \to \mathcal{I}_{S \cap Z, S}(t) \to 0 \,. \tag{1}$$

Hence $h^1(\mathbf{P}^3, \mathcal{I}_Z(t)) \leq h^1(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_S(Z)}(t - x)) + h^1(S, \mathcal{I}_{S \cap Z, S}(t))$. Since $C \cap \mathrm{Sing}(S) = \emptyset$, then $C^{(k-1)} \cap S$ is the effective Cartier divisor kC of S. Fix $P \in C$ and take a germ A at P of a surface transversal to C at P. Since both C and S are smooth at P, a local check using the slice A show that $\mathrm{Res}_S(C^{(k-1)}) = C^{(k-2)}$ (with the convention $C^{(-1)} = \emptyset$). Hence

$$h^1(\mathbf{P}^3, \mathcal{I}_{C^{(k-1)}}(t)) \leq h^1(\mathbf{P}^3, \mathcal{I}_{C^{(k-2)}}(t - x)) + h^1(S, \mathcal{O}_S(t)(-kC)) \,. \tag{2}$$

Proof of Theorem 1. Fix integers d, g, k, t as in the statement of Theorem 1. It is well-known (see [2] for more) that $h^1(\mathbf{P}^3, \mathcal{I}_T(t)) = 0$ for every $t \geq d-3$ and every smooth degree d curve $T \subset \mathbf{P}^3$ such that $p_a(T) > 0$. Hence by Remark 1 applied to the integer $x = 4$, to prove Theorem 1 for the quadruple (d, g, k, t) it is sufficient to prove the existence of a smooth quartic surface $S \subset \mathbf{P}^3$ (i.e. a smooth $K3$ surface) and an integral curve $C \subset S$ with degree d and genus g such that $h^1(S, \mathcal{O}_S((t - 4z) - (k - z)C)) = 0$ for all $0 \leq z \leq k - 1$. By [5] or [4], Theorem 4.6, there is a smooth degre 4 surface $S \subset \mathbf{P}^3$ and a smooth and connected curve $C_0 \subset \mathbf{P}^3$ with $\deg(C_0) = d_0$ and $p_a(C_0) = g_0$. Call H the hyperplane divisor of S. By [1], Proposition 2.2, there is a smooth and connected $C \in |C_0 + rH|$ with degree d and genus g. Hence $\mathcal{O}_S(y - (k-z)C) \cong \mathcal{O}_S((y - r(k-z))H - (k-z)C_0)$. Since the ideal sheaf of $C_0 \subset \mathbf{P}^3$ is spanned by forms of degree d_0 ([2]), $\mathcal{O}_S(yH-(k-z)C_0)$ is spanned for every $y \geq (k-z)d_0$. Since S is a $K3$ surface, we get $h^1(S, \mathcal{O}_S(yH-(k-z)C_0)) = 0$ for every $y \geq (k-z)d_0$, concluding the proof. □

Remark 2. It is very easy (using Remark 1) to give an upper bound for the index of regularity of powers of the ideal sheaf of any smooth curve contained in a smooth quadric surface. By [3] it would be very nice to check the case of curves contained in a smooth cubic surface. However, this case seems to be numerically very complicated.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, N. Chiarli, S. Greco, On the existence of k-normal curves of given degree in projective spaces, *Collectanea Math.*, **55**, No. 3 (2004), 279-287.

[2] L. Gruson, R. Lazarsfeld, C. Peskine, On a theorem of Castelnuovo, and the equations defining space curves, *Invent. Math.*, **72**, No. 3 (1983), 491-506.

[3] L. Gruson, Ch. Peskine, Courbes de l'espace projectif, II, *Ann. Sci. École Norm. Sup.*, **15**, No. 4 (1982), 401-418.

[4] A.L. Knutsen, Smooth curves on projective $K3$ surfaces, *Math. Scand.*, **90** (2000), 215-231.

[5] S. Mori, On degree and genera of curves on smooth quartic surfaces in $\mathbf{P}^3$, *Nagoya Math. J.*, **96** (1984), 127-132.

[6] P. Vermeire, On the regularity of powers of ideal sheaves, *Compositio Math.*, **131**, No. 2 (2002), 161-172.

[7] J. Wahl, On the cohomology of the square of an ideal sheaf, *J. Algebraic Geom.*, **6**, No. 3 (1997), 481-511.