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STABLE VECTOR BUNDLES ON CURVES: SYMMETRIC AND ANTI-SYMMETRIC MORPHISMS

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Abstract: Let C be a smooth curve of genus $g \geq 2$. For any vector bundle E on C and any $L \in \mathrm{Pic}(C)$ the involution $\sigma : A \otimes A \to A \otimes A$ defined by $\sigma(u \otimes v) = v \otimes u$ defines an involution on the vector space $W := H^0(C, \mathrm{Hom}(E \otimes E, L))$. Let W_+ (resp. W_-) its invariant (resp. anti-invariant) subspace. Here we compute $\dim(W_+)$ and $\dim(W_-)$ for all integers r, d, x such that $r \geq 2$, $x \geq g + 2d/r$ and $d \equiv 0 \pmod r$ when E is a general stable vector bundle with rank r and degree d and L is a general line bundle of degree x.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) \neq 2$. Let $M(n,n)$ denote the set of all $n \times n$ matrices with entries in $\mathbb{K}$. Set $\mathrm{Sym}(n) := \{A \in M(n,n) : A =^t A\}$ (the set of all symmetric matrices) and $\mathrm{Alt}(n) : \{A \in M(n,n) : A = -^t A\}$ (the set of all anti-symmetric matrices). Since $\mathrm{char}(\mathbb{K}) \neq 2$, we have $\mathrm{Sym}(n) + \mathrm{Alt}(n) = M(n,n)$ and $\mathrm{Sym}(n) \cap \mathrm{Alt}(n) = \{0\}$. Fix a linear subspace V of $M(n,n)$. What are the integers $\dim(V \cap \mathrm{Sym}(n))$ and $\dim(V \cap \mathrm{Alt}(n))$? Of course, in this naive form the question is meaningless, because for suitable

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V's the pair $(\dim(V \cap \mathrm{Sym}(n)), \dim(V \cap \mathrm{Alt}(n)))$ may be an arbitrary pair of integers (a, b) such that $0 \le a \le n(n+1)/2$, $0 \le b \le n(n-1)/2$ and $a + b \le \dim(V)$. Nevertheless, if we have an interesting (e.g. a geometrically significant V) it may be interesting to answer this question. We are interested in linear subspaces V such that $V \cap \mathrm{Sym}(n) + V \cap \mathrm{Alt}(n) = V$, i.e. we only consider linear subspaces which are invariant for the transposition. Our V's arise from the theory of stable vector bundles on a smooth projective curve. Let C be a smooth and connected projective curve of genus $g \ge 2$ over $\mathbb{K}$. Fix an integer $r \ge 2$, a rank r vector bundle E on C and $L \in \mathrm{Pic}(C)$. Set $W := H^0(C, \mathrm{Hom}(E \otimes E, L))$. Hence W is the vector space of all bilinear maps from $E \oplus E$ to L. Let $\sigma : A \otimes A \to A \otimes A$ be the transposition. Since σ acts on W, we may speak about symmetric and anti-symmetric elements of W. Let W_+ (resp. W_-) be the corrisponding vector spaces. The integers $\dim(W_+)$ and $\dim(W_-)$ may depend from E, even for fixed $\dim(W)$. Here we show the existence of an interesting case in which these integers are computable and "essentially" independent from E and L (if we fix the integers $\mathrm{rank}(E)$, $\deg(E)$ and $\deg(L)$). Let $M(C, r, d)$ denote the set of all vector bundles on C with rank r and degree d. $M(C, r, d)$ is a non-empty irreducible smooth variety of dimension $r^2(g-1)+1$. Hence it is reasonable to consider the properties of a general $E \in M(C, r, d)$ and these general stable bundles will be the vector bundles which we will consider in this paper. The aim is to prove the following result.

Theorem 1. *Fix integers r, d, x such that $r \ge 2$, $x \ge g + 2d/r$ and $d \equiv 0 \pmod r$ and a general pair $(E, L) \in M(C, r, d) \times \mathrm{Pic}^x(C)$. Set $W := H^0(C, \mathrm{Hom}(E \otimes E, L))$. Then $\dim(W) = r^2(x+1-g) - 2rd$, $\dim(W_+) = r(r+1)(x+1-g)/2 - (r+1)d/r$ and $\dim(W_-) = r(r-1)(x+1-g)/2 - (r-1)d/r$.*

To explain the numerical assumptions in Theorem 1 we recall that for general E, L we have $W = \{0\}$ if $x \le g - 1 + 2d/r$ (copy the proof of the well-known Lemma 1 below).

2. The Proofs

We will first compute the integer $\dim(W)$. We stress that this integer is well-known, i.e. that the following lemma is well-known.

Lemma 1. *Fix integers r, d, x such that $r \ge 2$, $x \ge g + 2d/r$ and $d \equiv 0 \pmod r$ and a general pair $(E, L) \in M(C, r, d) \times \mathrm{Pic}^x(C)$. Then $h^0(C, \mathrm{Hom}(E \otimes E, L)) = r^2(x+1-g) - 2rd$.*

Proof. Fix $A \in \mathrm{Pic}^{d/r}(C)$ and set $B := A^{\oplus r}$. By Riemann-Roch the statement of the lemma is equivalent to $h^1(C, \mathrm{Hom}(E \otimes E, L)) = 0$. Hence we may apply the semicontinuity theorem for cohomology groups ([1], Theorem III.12.8). Every vector bundle on C is a flat limit of a family of stable vector bundles on C with the same degree and rank (see [3], Proposition 2.2, in characteristic zero or [2], Corollary 2.2, for a characteristic free easy proof). In particular, this is true for the vector bundle B. Hence by semicontinuity it is sufficient to prove $h^1(C, \mathrm{Hom}(B \otimes B, L)) = 0$. Since $\mathrm{Hom}(E \otimes E, L) \cong (A^* \otimes L)^{\oplus r^2}$ and $A^* \otimes L$ is a general line bundle of degree $x - 2d/r \geq g$, this vanishing is obvious. □

Remark 1. Let S be a reduced algebraic scheme and $\{V_s\}_{s \in S}$ an algebraic family of linear subspaces of $M(n, n)$ parametrized by S. Hence the family of integers $\{\dim(V_s)\}_{s \in S}$ is locally constant. Assume that each V_s is invariant for action of the trasposition, i.e. assume $V_s \cap \mathrm{Sym}(n) + V_s \cap \mathrm{Alt}(n) = V_s$ for every $s \in S$. Since the functions $\dim(V \cap \mathrm{Sym}(n))$ and $\dim(V \cap \mathrm{Alt}(n))$ are upper semicontinuous and their sum is locally constant, both functions are locally constant.

Proof of Theorem 1. Fix a general pair $(A, L) \in \mathrm{Pic}^{d/r}(C) \times \mathrm{Pic}^x(C)$ and set $B := A^{\oplus r}$. As in the proof of Lemma 1 B is a flat limit of a family of elements of $M(C, r, d)$ and $h^0(C, \mathrm{Hom}(B \otimes B, L)) = r^2(x + 1 - g) - 2rd$. Set $M := H^0(C, \mathrm{Hom}(B \otimes B, L))$. By Remark 1 it is sufficient to prove that $\dim(M_+) = r(r+1)(x+1-g)/2 - (r+1)d/r$ and $\dim(M_-) = r(r-1)(x+1-g)/2 - (r-1)d/r$. Fix a general $D \subset C$ with $\mathrm{card}(D) = x - g - 2d/r$. Hence $L(-D) \otimes A^*$ may be considered as a general line bundle on C with degree $g - 1$. Thus $h^0(C, L(-D) \otimes A^*) = h^1(C, L(-D) \otimes A^*) = 0$. Hence the restriction map $M \to L \otimes B^*|D$ is bijective. Since the transposition σ commutes with the restriction to a fiber, we reduce the computation of $\dim(M_+)$ and $\dim(M_-)$ to the corresponding computation of a finite direct sum of the vector spaces $M(r, r)$ with the usual transposition, concluding the proof. □

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