ON THE EXISTENCE OF EXTREMAL SOLUTIONS
OF PHI-LAPLACIAN INITIAL AND
BOUNDARY VALUE PROBLEMS

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Abstract: In this paper we apply fixed point results for mappings in partially
ordered function spaces to derive existence results for extremal solutions of some
phi-Laplacian initial and boundary value problems. The considered problems
can be singular, functional, nonlocal and discontinuous. Concrete examples are
also solved.

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1. Introduction

In this paper we apply fixed point results presented in [1, 4] to derive existence
results for phi-Laplacian initial- and boundary value problems. The considered
problems include many kinds of special types. For instance:

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– the differential equations may be singular;
– both the differential equations and the initial or boundary conditions may depend functionally on the unknown function and/or on its first derivative;
– both the differential equations and the initial or boundary conditions may contain discontinuous nonlinearities;
– problems on infinite intervals are also included.
Concrete examples are solved to illustrate the obtained results.

2. Existence Results for First Order IVP’s

In this section we study the first order initial value problem (IVP)

\[
\begin{align*}
\frac{d}{dt}(p(t)\phi(u(t))) &= f(t, u) \text{ for almost every (a.e.) } t \in J := (a, b), \\
\lim_{t \to a^+} p(t)\phi(u(t)) &= c(u),
\end{align*}
\]

(2.1)

where \(-\infty \leq a < b \leq \infty\), \(p \in C(J)\), \(\phi : \mathbb{R} \to \mathbb{R}\), \(f : J \times C(J) \to \mathbb{R}\) and \(c : C(J) \to \mathbb{R}\).

We are looking for solutions of (2.1) from the set

\[ S := \{ u \in C(J) \mid p \cdot (\phi \circ u) \text{ is locally absolutely continuous}\}. \]

(2.2)

Denote

\[ X := \{ h \in L^1_{loc}(J) \mid \int_{a^+}^{s} h(t) \, dt := \lim_{r \downarrow a^+} \int_{r}^{s} h(t) \, dt \text{ is finite for an } s \in J\}. \]

(2.3)

Assuming that \(X\) is ordered a.e. pointwise, and that \(C(J)\) and \(S\) are ordered pointwise, we shall show that the IVP (2.1) has extremal solutions in \(S\) if the functions \(p, \phi, f\) and \(c\) satisfy the following hypotheses:

\((p)\) \(p : J \to \mathbb{R}\) is continuous and positive-valued.

\((\phi)\) \(\phi\) is an increasing homeomorphism from an open interval \(I\) of \(\mathbb{R}\) onto \(\mathbb{R}\).

\((fa)\) \(f(\cdot, u)\) is Lebesgue measurable and \(X \ni h_- \leq f(\cdot, u) \leq h_+ \in X\) for all \(u \in C(J)\).

\((fb)\) \(f(\cdot, u) \leq f(\cdot, v)\) whenever \(u, v \in C(J)\), \(u \leq v\).
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(c) There exist $c_\pm \in \mathbb{R}$ such that $c_- \leq c(u) \leq c(v) \leq c_+$ for all $u, v \in C(J)$, $u \leq v$.

We shall first convert the IVP (2.1) to an integral equation.

**Lemma 2.1.** Assume that the hypotheses ($\phi$) and (p) hold. If $u \in S$ and $f(\cdot, u) \in X$, then $u$ is a solution of the IVP (2.1) if and only if

$$u(t) = \phi^{-1} \left( \frac{1}{p(t)} [c(u) + \int_{a+}^{t} f(s, u) \, ds] \right), \quad t \in J. \quad (2.4)$$

**Proof.** Assume that $u \in S$ is a solution of (2.1), and that $f(\cdot, u) \in X$. The differential equation of (2.1) and the definition (2.2) of $S$ ensure that

$$\int_{s}^{r} f(t, u) \, dt = \int_{s}^{r} \frac{d}{dt}(p(t)\phi(u(t))) \, dt = p(s)\phi(u(s)) - p(r)\phi(u(r)), \quad a < r \leq s < b.$$

This result and the initial condition of (2.1) imply that (2.4) holds.

The converse part of the proof is trivial. □

The following fixed point result is a consequence of [1], Theorem A.2.1.

**Lemma 2.2.** Assume that $G : C(J) \to C(J)$ is increasing, i.e., $Gu \leq Gv$ whenever $u \leq v$, that the range ran $G$ of $G$ is order bounded, and that each well-ordered chain of ran $G$ has a supremum in $C(J)$ and each inversely well-ordered chain has an infimum in $C(J)$. Then $G$ has least and greatest fixed points, and they are increasing with respect to $G$.

Now we are ready to prove our main existence result for the IVP (2.1).

**Theorem 2.1.** Assume that the hypotheses ($\phi$), (p), (fa), (fb) and (c) hold. Then the IVP (2.1) has least and greatest solutions in $S$, and they are increasing with respect to $f$ and $c$.

**Proof.** The given hypotheses imply that the equation

$$Gu(t) := \phi^{-1} \left( \frac{1}{p(t)} [c_- + \int_{a+}^{t} f(s, u) \, ds] \right), \quad t \in J, \quad (2.5)$$

defines a mapping $G : C(J) \to C(J)$. The relations

$$\begin{cases} u(t) := \phi^{-1} \left( \frac{1}{p(t)} [c_- + \int_{a+}^{t} h_-(s) \, ds] \right), & t \in J, \\ \overline{u}(t) := \phi^{-1} \left( \frac{1}{p(t)} [c_+ + \int_{a+}^{t} h_+(s) \, ds] \right), & t \in J, \end{cases} \quad (2.6)$$

Now we are ready to prove our main existence result for the IVP (2.1).
define functions $u$ and $\bar{u}$ of $S$ for which $u \leq \bar{u}$. Moreover, it is easy to see that $u \leq Gu \leq Gv \leq \bar{u}$ for all $u, v \in C(J)$, $u \leq v$. Thus $G$ is increasing and its range is order-bounded.

Let $W$ be a well-ordered chain in ran $G$. The given hypotheses and the definitions of $\leq$ and $G$ imply that $W$ is a well-ordered and equicontinuous chain in $C(J)$, and that $u \leq u' \leq u''$ for each $u \in W$. It then follows from [4], Proposition 1.3.8, that $u = \sup W$ exists in $C(J)$. Similarly one can show that each inversely well-ordered chain of ran $G$ has an infimum in $C(J)$.

The above proof shows that the operator $G$ defined by (2.5) satisfies the hypotheses of Lemma 2.2, whence $G$ has a least fixed point $u^*$ and a greatest fixed point $u^*$. According to Lemma 2.1 $u^*$ and $u^*$ are least and greatest solutions of the IVP (2.1). The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition of $G$.

As a special case we obtain an existence result for the IVP

$$\left\{ \begin{array}{lcl}
\frac{d}{dt}(p(t)\phi(u(t))) &=& g(t, u(t)) \text{ for a.e. } t \in J, \\
\lim_{t \to a^+} p(t)\phi(u(t)) &=& c.
\end{array} \right. \quad (2.7)$$

Proposition 2.1. Let the hypotheses $(\phi)$ and $(p)$ hold, and let $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy the following hypotheses:

$(ga)$ $g(\cdot, u(\cdot))$ is Lebesgue measurable and $X \ni h_- \leq g(\cdot, u(\cdot)) \leq h_+ \in X$ for all $u \in C(J)$.

$(gb)$ $g(t, x) \leq g(t, y)$ for a.e. $t \in J$ and whenever $x \leq y$ in $\mathbb{R}$.

Then the IVP (2.7) has for each choice of $c \in \mathbb{R}$ least and greatest solutions in $S$. Moreover, these solutions are increasing with respect to $g$ and $c$.

Proof. If $c \in \mathbb{R}$, the IVP (2.6) is reduced to (2.1) when we define

$$\left\{ \begin{array}{lcl}
f(t, u) &=& g(t, u(t)), \quad t \in J, \; u \in C(J), \\
c(u) &\equiv& c, \quad u \in C(J).
\end{array} \right. \quad (2.8)$$

The hypotheses (ga) and (gb) imply that $f$ satisfies the hypotheses (fa) and (fb). The hypothesis (c) is also valid, whence (2.1), with $f$ and $c$ defined by (2.8), and hence also (2.7), has by Theorem 2.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 2.1.

If we replace the hypothesis (fa) by the following hypothesis:
(fc) $f(\cdot, u) \in X$ for all $u \in C(J)$, and there exist $u_{\pm} \in C(J)$ such that $u_- \leq u_+, u_- \leq Gu_-$ and $Gu_+ \leq u_+$, where $G$ is defined by (2.5), we get the following result.

**Corollary 2.1.** Assume that the hypotheses $(\phi), (p), (fb), (fc)$ and $(c)$ hold. Then the IVP (2.1) has a least and a greatest solution in $\{u \in S \mid u_- \leq u \leq u_+\}$.

**Remarks 2.1.** If $\lim_{t \to a^+} p(t) = 0$, the differential operator $\frac{d}{dt}(p(t)\phi(u(t)))$ in (2.1) is singular. An example of a function $\phi$ with property $(\phi)$ is

$$\phi(x) = \frac{x}{\sqrt{1-x^2}}, \quad x \in (-1, 1),$$

arising in relativistic dynamics. In this case the operator $G$ given by (2.5) can be rewritten as

$$Gu(t) = \frac{c(u) + \int_{a_+}^{t} f(s, u) \, ds}{\sqrt{p^2(t) + (c(u) + \int_{a_+}^{t} f(s, u) \, ds)^2}}, \quad t \in J. \quad (2.9)$$

This formula shows that $-1 \leq Gu(t) \leq 1$ whenever the right hand side of (2.9) is defined. Thus we have the following result.

**Proposition 2.2.** Assume that $p \in C(J)$ is positive-valued, and that $f(\cdot, u) \leq f(\cdot, v)$ in $X$ and $c(u) \leq c(v)$ in $\mathbb{R}$ whenever $u \leq v$ in $C(J)$. Then the IVP

$$\frac{d}{dt}\left(\frac{p(t)u(t)}{\sqrt{1-u(t)^2}}\right) = f(t, u) \quad \text{a.e. in } J = (a, b),$$

$$\lim_{t \to a^+} \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} = c(u) \quad (2.10)$$

has least and greatest solutions in $S$.

As a consequence of Proposition 2.2 we obtain an existence result also for a periodic boundary value problem.

**Corollary 2.2.** Let $p$ and $f$ satisfy the hypotheses of Proposition 2.2. Then for each choice of $t_1, t_2 \in J$, $t_1 < t_2$, the periodic boundary value problem

$$\frac{d}{dt}\left(\frac{p(t)u(t)}{\sqrt{1-u(t)^2}}\right) = f(t, u) \quad \text{a.e. in } [t_1, t_2], \quad u(t_1) = u(t_2), \quad (2.11)$$

has least and greatest solutions.
Proof. The asserted result follows from Proposition 2.2 when we replace \( a \) by \( t_1 \) and \( c(u) \) by \( \frac{p(t_1)u(t_2)}{\sqrt{1-u(t_2)^2}} \) in (2.10).

Example 2.1. Consider the IVP
\[
\begin{aligned}
\frac{d}{dt}\left( \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} \right) &= h(t) + K\left[ q(t) \int_{1}^{2} u(s) \, ds \right], \quad \text{a.e. in } J := (0, \infty), \\
\lim_{t \to 0^+} \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} &= c \cdot \frac{p(1)u(1)}{\sqrt{1-u(1)^2}},
\end{aligned}
\]

(2.12)

where \( p \in C(J) \), \( p(t) > 0 \) for \( t \in J \), \( q \in L^1(J) \), \( h \in X \), \( K, c \geq 0 \), and \([z]\) denotes the greatest integer \( \leq z \). Problem (2.12) is of the form (2.10) with
\[
c(u) = c \cdot \frac{p(1)u(1)}{\sqrt{1-u(1)^2}} \quad \text{and} \quad f(t, u) = h(t) + K\left[ q(t) \int_{1}^{2} u(s) \, ds \right], \quad t \in J. \quad (2.13)
\]

It is easy to see that the hypotheses of Proposition 2.2 hold, whence (2.12) has least and greatest solutions.

Remark 2.2. If \( h(t) = \frac{1}{t} \sin \frac{1}{t} \), \( t \in J = (0, \infty) \), then \( h \) and the functions \( f(\cdot, u) \) and \( h_\pm \) defined by (2.12) and (2.13) belong to \( X \), but not to \( L^1((0, T)) \) for any \( T > 0 \).

Example 2.2. The singular IVP
\[
\frac{d}{dt}\left( \frac{tu(t)}{\sqrt{1-u(t)^2}} \right) = \frac{1}{10} (t + 10 \int_{1}^{2} u(s) \, ds), \quad \text{a.e. in } (0, \infty),
\]

\[
\lim_{t \to 0^+} \frac{tu(t)}{\sqrt{1-u(t)^2}} = 0 \quad (2.14)
\]

is a special case of (2.12) when \( p(t) = t \), \( q(t) \equiv 10 \), \( h(t) = \frac{1}{10} \), \( K = \frac{1}{10} \) and \( c = 0 \). Thus (2.14) has extremal solutions. To determine them, notice first that we can choose \( c_\pm = 0 \) in (2.6). Because solutions of (2.14) are fixed points of \( G \) of the form (2.9), we may assume that \( |\int_{1}^{2} u(s) \, ds| \leq 1 \). Thus we can choose \( h_\pm(t) = \frac{1}{10} \pm 1 \) in (2.6). With these choices the functions \( u \) and \( \overline{u} \) defined by (2.6) can be calculated, and one obtains
\[
u(t) = \frac{t - 20}{\sqrt{t^2 - 400t + 800}}, \quad \overline{u}(t) = \frac{t + 20}{\sqrt{t^2 + 400t + 800}}.
\]

The first elements of the chains of \( G \)-iterations needed to prove Theorem 2.1 are iterations \( G^n u \) and \( G^n \overline{u} \), \( n = 0, 1, \ldots \), where \( G \) is defined by (2.9) (cf. [1],...
Remark A.2.1). Calculating the iterations $G^nu$, it turns out that $G^4u = G^5u$. Thus $u^* = G^4u$ is a least solution of (2.14) by [4], Corollary 1.1.2. Similarly, one can show that $u^* = G^5u$ is a greatest solution of (2.14). The exact expressions of these solutions are

$$u^*(t) = \frac{t - 8}{\sqrt{t^2 - 16t + 464}}, \quad u^*(t) = \frac{t + 8}{\sqrt{t^2 + 16t + 464}}.$$ 

To obtain other solutions we can apply Corollary 2.1. For instance, the hypotheses of Corollary 2.1 are valid when

$$u(t) = \frac{t \pm 7}{\sqrt{t^2 + 14t + 449}}.$$ 

Applying the iteration method described above one can show that the extremal solutions of (2.5) in $\{u \in S \mid u_\pm \leq u \leq u_\pm\}$ are

$$u(t) = \frac{t \pm 6}{\sqrt{t^2 + 12t + 436}}.$$ 

By suitable choices of the functions $u_\pm$ one can show that all functions

$$u(t) = \frac{t \pm 2k}{\sqrt{(t \pm 2k)^2 + 20^2}}, \quad k = 0, 1, 2, 3, 4,$$

are solutions of (2.14).

## 3. Existence Results for Second Order IVP's

Next we study the second order phi-Laplacian initial value problem (IVP)

$$\begin{cases} \frac{d}{dt}(p(t)\phi(u'(t))) = f(t, u, u') \text{ for a.e. } t \in J := (a, b), \\ \lim_{t \to a^+} p(t)\phi(u'(t)) = c(u, u'), \quad \lim_{t \to a^+} u(t) = d(u, u'), \end{cases} \quad (3.1)$$

where $-\infty \leq a < b \leq \infty$, $p \in C(J)$, $\phi : \mathbb{R} \to \mathbb{R}$, $f : J \times C(J) \times C(J) \to \mathbb{R}$ and $c, d : C(J) \times C(J) \to \mathbb{R}$.

We are now looking for solutions of (3.1) from the set

$$Y := \{u \in C^1(J) \mid p \cdot (\phi \circ u') \text{ is locally absolutely continuous}\}. \quad (3.2)$$
Denote, as in Section 2,

\[ X := \{ h \in L^1_{\text{loc}}(J) \mid \int_{a^+}^{s} h(t) \, dt := \lim_{\nu \to a^+} \int_{\nu}^{s} h(t) \, dt \text{ is finite for an } s \in J \}. \quad (3.3) \]

Assuming that \( X \) is ordered a.e. pointwise, and that \( C(J) \) and \( Y \) are ordered pointwise, we shall show that the IVP (3.1) has extremal solutions in \( Y \) if the functions \( p, \phi, f, c \) and \( d \) satisfy the following hypotheses:

- \((\phi)\) \( \phi \) is an increasing homeomorphism from an open interval \( I \) of \( \mathbb{R} \) onto \( \mathbb{R} \).
- \((p\phi)\) \( p : J \to \mathbb{R} \) is continuous and positive-valued, and \( \phi^{-1}(\frac{K}{p(r)}) \in X \), for all \( K \in \mathbb{R} \).
- \((f0)\) \( f(\cdot, u, u') \) is Lebesgue measurable and \( X \ni h_- \leq f(\cdot, u, u') \leq h_+ \in X \) for all \( u \in C^1(J) \).
- \((f1)\) \( f(\cdot, u, u') \leq f(\cdot, v, v') \) whenever \( u, v \in C^1(J), u \leq v \) and \( u' \leq v' \).
- \((c0)\) \( c_+ \in \mathbb{R}, \text{ and } c_- \leq c(u, u') \leq c(v, v') \leq c_+ \text{ if } u, v \in C^1(J), u \leq v \) and \( u' \leq v' \).
- \((d0)\) \( d_+ \in \mathbb{R}, \text{ and } d_- \leq d(u, u') \leq d(v, v') \leq d_+ \text{ if } u, v \in C^1(J), u \leq v \) and \( u' \leq v' \).

Our first task is to convert the IVP (3.1) to an integral equation.

**Lemma 3.1.** Assume that the hypotheses \((\phi)\) and \((p\phi)\) hold. If \( u \in Y \) and \( f(\cdot, u, u) \in X \), then \( u \) is a solution of the IVP (3.1) if and only if

\[ u(t) = d(u, u') + \int_{a^+}^{t} \phi^{-1} \left( \frac{1}{p(s)} [c(u, u') + \int_{a^+}^{s} f(x, u, u') \, dx] \right) \, ds, \quad t \in J. \quad (3.4) \]

**Proof.** Assume that \( u \in Y \) is a solution of (3.1), and that \( f(\cdot, u, u') \in X \). The differential equation of (3.1) and the definition (3.2) of \( Y \) ensure that

\[ \int_{r}^{s} f(t, u, u') \, dt = \int_{r}^{s} \frac{d}{dt}(p(t)\phi(u'(t))) \, dt = p(s)\phi(u'(s)) - p(r)\phi(u'(r)), \]

when \( a < r \leq s < b \). This result and the first initial condition of (3.1) imply that

\[ u'(s) = \phi^{-1} \left( \frac{1}{p(s)} [c(u, u') + \int_{a^+}^{s} f(x, u, u') \, dx] \right), \quad s \in J. \quad (3.5) \]
In view of the hypotheses \((\phi)\) and \((p\phi)\) we can integrate (3.5) and apply the second initial condition of (3.1) to obtain (3.4).

The converse part of the proof is trivial.

**Lemma 3.2.** Let \(W\) be an equicontinuous and pointwise order-bounded subset of \(C(J)\).

a) If \(W\) is well-ordered, there exists an increasing sequence in \(W\) which converges to \(\sup W\) locally uniformly in \(J\).

b) If \(W\) is inversely well-ordered, there exists a decreasing sequence in \(W\) which converges to \(\inf W\), locally uniformly in \(J\).

**Proof.** a) Assume first that \(W\) is well-ordered, i.e. each nonempty subset of \(W\) has a minimum. Since \(W\) is pointwise order-bounded, its increasing sequences converge pointwise. It then follows from [3], Proposition 5 that there exists an increasing sequence \((u_n)_{n=1}^{\infty}\) in \(W\) which converges pointwise to \(u = \sup W\). Because \(u\) belongs to \(C(J)\), the convergence is locally uniform by Dini’s Theorem.

b) If \(W\) is inversely well-ordered, then \(-W\) is well-ordered, and the application of the above reasoning to \(-W\) implies the assertion in the case b).

Define a partial ordering in \(C^1(J)\) by

\[
 u \preceq v \text{ iff } u \leq v \text{ and } u' \leq v'.
\] (3.6)

The following fixed point result is a consequence of [4], Theorem 1.2.1 and Proposition 1.2.1.

**Lemma 3.3.** Assume that \(G : C^1(J) \to C^1(J)\) is increasing, that the range \(\text{ran} \ G\) of \(G\) is order bounded, and that each well-ordered chain of \(\text{ran} \ G\) has a supremum and each inversely well-ordered chain of \(\text{ran} \ G\) has an infimum. Then \(G\) has least and greatest fixed points, and they are increasing with respect to \(G\).

Now we are ready to prove our main existence result for the IVP (3.1).

**Theorem 3.1.** Assume that the hypotheses \((\phi), (p\phi), (f0), (f1), (c0)\) and \((d0)\) hold. Then the IVP (3.1) has least and greatest solutions in \(Y\), and they are increasing with respect to \(f, c\) and \(d\).
Proof. The given hypotheses imply that the equation
\[ Gu(t) := d(u, u') + \int_{a+}^{t} \left( \frac{1}{p(s)} [c(u, u') + \int_{a+}^{s} f(x, u, u') \, dx] \right) \, ds, \quad t \in J, \]  
(3.7)
defines a mapping \( G : C^1(J) \to C^1(J) \). The relations
\[
\begin{align*}
\underline{u}(t) & := d_- + \int_{a+}^{t} \phi^{-1} \left( \frac{1}{p(s)} [c_- + \int_{a+}^{s} h_-(x) \, dx] \right) \, ds, \quad t \in J, \\
\overline{u}(t) & := d_+ + \int_{a+}^{t} \phi^{-1} \left( \frac{1}{p(s)} [c_+ + \int_{a+}^{s} h_+(x) \, dx] \right) \, ds, \quad t \in J
\end{align*}
\]
(3.8)
define functions \( \underline{u}, \overline{u} \in Y \) for which \( \underline{u} \leq \overline{u} \). Moreover, it is easy to see that \( \underline{u} \leq Gu \leq \overline{u} \leq \overline{u} \) for all \( u, v \in C^1(J), u \leq v \). Thus \( G \) is increasing and its range is order-bounded.

Let \( W \) be a well-ordered chain in ran \( G \). By the definitions of \( \leq \) and \( G \) this implies that the set \( V = \{ v' \mid v \in W \} \) is a well-ordered chain in \( C(J) \), and \( u' \leq v' \leq \overline{u} \) for each \( v' \in V \). In particular, \( \underline{u}'(t) \leq v'(t) \leq \overline{u}'(t) \) for all \( t \in J \) and \( v' \in V \). The given hypotheses ensure also that \( V \) is equicontinuous. Thus there exists by Lemma 3.2.a) an increasing sequence \( (v_n') \) of \( V \) which converges locally uniformly in \( J \) to \( w = \sup V \).

The given hypotheses and the definitions of \( \leq \) and \( G \) imply also that \( W \) is a well-ordered and equicontinuous chain in \( C(J) \), and that \( \underline{u} \leq v \leq \overline{u} \) for each \( u \in W \). It then follows from Lemma 3.2.a) that \( u = \sup W \) exists in \( C(J) \), and there exists an increasing sequence \( (u_n) \) in \( W \) which converges locally uniformly in \( J \) to \( u \). Denoting \( w_n = \max\{u_n, v_n\}, n \in \mathbb{N} \), where maximum is taken with respect to \( \leq \), then \( u_n \leq w_n \leq u \) in \( C(J) \) and \( v_n' \leq w_n' \leq w \) in \( C(J) \). These relations and the above proof imply that \( w_n \to u \) and \( w_n' \to w \) locally uniformly in \( J \). Thus \( w(t) = u'(t) \) in \( J \), by [2], (8.6.3), so that \( u \in C^1(J) \). This result and the definitions of \( u \) and \( w \) imply that \( u = \sup W \) in \( C^1(J), \leq \). Similarly one can show that each inversely well-ordered chain of ran \( G \) has an infimum.

The above proof shows that the operator \( G \) defined by (3.4) satisfies the hypotheses of Lemma 3.3, whence \( G \) has a least fixed point \( u_* \) and a greatest fixed point \( u^* \). According to Lemma 3.1 \( u_* \) and \( u^* \) are least and greatest solutions of the IVP (3.1). The last assertion is an easy consequence of the last assertion of Lemma 3.3 and the definition (3.7) of \( G \).

If we replace the hypothesis \( (f_0) \) by the following hypothesis:

(\( f(\cdot, u, u') \in X \) for all \( u \in C^1(J) \), and there exist \( u_{\pm} \in C^1(J) \) such that \( u_- \leq u_+ \), \( u_- \leq Gu_- \) and \( Gu_+ \leq u_+ \), where \( G \) is defined by (3.7),

we get the following result.

**Proposition 3.1.** Assume that the hypotheses \( \phi \), \((p\phi)\), \((f1)\), \((f2)\), \((c0)\) and \((d0)\) hold. Then the IVP (3.1) has a least and a greatest solution in \( \{ u \in Y \mid u_+ \leq u \leq u_+ \} \).

As a special case we obtain an existence result for the IVP

\[
\begin{cases}
\frac{d}{dt}(p(t)\phi(u'(t))) = g(t, u(t), u'(t)) \text{ for a.e. } t \in J, \\
\lim_{t \to t_0^+} p(t)\phi(u'(t)) = c, \quad \lim_{t \to t_0^+} u(t) = d.
\end{cases}
\] (3.9)

**Corollary 3.1.** Let the hypotheses \( \phi \) and \((p\phi)\) hold, and let \( g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfy the following hypotheses:

\( (g0) \) \( g(\cdot, u(\cdot), u'(\cdot)) \) is Lebesgue measurable and \( h_- \leq g(\cdot, u(\cdot), u'(\cdot)) \leq h_+ \) for all \( u \in C^1(J) \) and for some \( h_\pm \in X \).

\( (g1) \) \( g(t, x, z) \leq g(t, y, w) \) for a.e. \( t \in J \) and whenever \( x \leq y \) and \( z \leq w \) in \( \mathbb{R} \).

Then the IVP (3.9) has for each choice of \( c, d \in \mathbb{R} \) least and greatest solutions in \( Y \). Moreover, these solutions are increasing with respect to \( g, c \) and \( d \).

**Proof.** If \( c, d \in \mathbb{R} \), the IVP (3.9) is reduced to (3.1) when we define

\[
\begin{cases}
f(t, u, u') = g(t, u(t), u'(t)), \quad t \in J, \ u \in C^1(J), \\
c(v, v') \equiv c, \quad v \in C^1(J), \quad d(v, v') \equiv d, \quad v \in C^1(J).
\end{cases}
\] (3.10)

The hypotheses \((g0)\) and \((g1)\) imply that \( f \) satisfies the hypotheses \((f0)\) and \((f1)\). The hypotheses \((c0)\) and \((d0)\) are also valid, whence (3.1), with \( f, c \) and \( d \) defined by (3.10), and hence also (3.9), has by Theorem 3.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 3.1. □

**Remarks 3.1.** If \( \lim_{t \to t_0^+} p(t) = 0 \), the differential operator \( \frac{d}{dt}(p(t)\phi(u'(t))) \) in (3.1) is a singular phi-Laplacian operator. A special case of it is the p-Laplacian operator with

\[\phi(x) = |x|^{p-2}x, \quad x \in (-\infty, \infty).\] (3.11)

The hypothesis \((p\phi)\) holds, for instance, if \( p(t) = t \) and \( 1 < p < 2 \). Another example of \( \phi \) is

\[\phi(x) = \frac{x}{\sqrt{1-x^2}}, \quad x \in (-1,1).\]
In this case the operator $G$ given by (3.7) can be rewritten as

$$Gu(t) = d(u, u') + \int_{a+}^{t} \frac{c(u, u') + \int_{a+}^{s} f(x, u, u') \, dx}{\sqrt{p^2(s) + (c(u, u') + \int_{a+}^{s} f(x, u, u') \, dx)^2}} \, ds,$$

whenever the right hand side of (3.12) is defined. Thus we have the following result.

**Corollary 3.2.** If $-\infty < a < b \leq \infty$, the IVP

$$\begin{cases} \frac{d}{dt}\left( \frac{p(t)u(t)}{\sqrt{1 - u'(t)^2}} \right) = f(t, u, u') & \text{a.e. in } J = (a, b), \\ \lim_{t \to a^+} \frac{p(t)u'(t)}{\sqrt{1 - u'(t)^2}} = c(u, u'), & u(a) = d(u, u'), \end{cases}$$

(3.13)

has least and greatest solutions if $p(t) > 0$ for all $t \in J$, if the hypothesis (d0) holds, and if $f(\cdot, u, u') \leq f(\cdot, v, v')$ in $X$ and $c(u, u') \leq c(v, v')$ in $\mathbb{R}$ whenever $u \leq v$ and $u' \leq v'$ in $C(J)$.

**Example 3.1.** Consider the IVP

$$\begin{cases} \frac{d}{dt}\left( \frac{p(t)u'(t)}{\sqrt{1 - u'(t)^2}} \right) = h(t) + K[q(t) \int_{1}^{2} (u(s) + u'(s)) \, ds] & \text{a.e. in } J := (0, \infty), \\ \lim_{t \to 0^+} \frac{p(t)u'(t)}{\sqrt{1 - u'(t)^2}} = c \cdot u'(1), & u(0) = \frac{[u(1) + u'(1)]}{1 + [u(1) + u'(1)]}, \end{cases}$$

(3.14)

where $p \in C(J)$, $\lim_{t \to 0^+} p(t) = 0$, $p(t) > 0$ for $t \in J$, $q \in L^1_+(J)$, $h \in X$, $c \geq 0$, and $[z]$ denotes the greatest integer $\leq z$. The problem (3.14) is of the form (3.13) with

$$\begin{cases} f(t, u, u') = h(t) + K[q(t) \int_{1}^{2} (u(s) + u'(s)) \, ds], & t \in J, \\ c(u, u') = c \cdot u'(1), & d(u, u') = \frac{[u(1) + u'(1)]}{1 + [u(1) + u'(1)]}. \end{cases}$$

(3.15)

It is easy to see that the hypotheses of Corollary 3.2 hold, whence the IVP (3.14) has least and greatest solutions.

**Example 3.2.** The singular IVP

$$\begin{cases} \frac{d}{dt}\left( \frac{tu'(t)}{\sqrt{1 - u'(t)^2}} \right) = \frac{t}{10} + \frac{1}{30} [10 \int_{1}^{2} (u(s) + u'(s)) \, ds] & \text{a.e. in } J := (0, \infty), \\ \lim_{t \to 0^+} \frac{tu'(t)}{\sqrt{1 - u'(t)^2}} = 0, & u(0) = \frac{[u(1) + u'(1)]}{1 + [u(1) + u'(1)]}. \end{cases}$$

(3.16)
is a special case of (3.14) when \( p(t) = t, \ q(t) \equiv 10, \ h(t) = \frac{t}{10}, \ K = \frac{1}{30} \) and \( c = 0 \). Thus (3.16) has extremal solutions, which are extremal fixed points of operator \( G \) of the form (3.12). The functions \( \underline{u} \) and \( \overline{u} \) defined by (3.8) can be calculated, and we can choose \( d_\pm = \pm 1, \ c_\pm = 0 \) and \( h_\pm (t) = \frac{t}{10} \pm 1 \), to obtain

\[
\underline{u}(t) = -1 - 20\sqrt{2} + \sqrt{t^2 - 40t + 800}, \quad \overline{u}(t) = 1 - 20\sqrt{2} + \sqrt{t^2 + 40t + 800}.
\]

Calculating the iterations \( G^n \underline{u} \), where \( G \) is defined by (3.12), it turns out that \( G^5 \underline{u} = G^6 \underline{u} \). Thus \( \underline{u}^* = G^5 \underline{u} \) is a least solution of (3.16) by [4], Corollary 1.1.2. Similarly, one can show that \( \overline{u}^* = G^5 \overline{u} = G^6 \overline{u} \) is a greatest solution of (3.16). The exact expressions of these solutions are

\[
\begin{align*}
\underline{u}^*(t) &= -\frac{2}{3} - \frac{2}{3}\sqrt{1069 + \frac{1}{3}\sqrt{9t^2 - 156t + 4276}}, \\
\overline{u}^*(t) &= \frac{1}{2} - \frac{2}{3}\sqrt{1261 + \frac{1}{3}\sqrt{9t^2 + 288t + 5044}}.
\end{align*}
\]

To obtain other solutions we can apply Proposition 3.1. For instance, the hypothesis (f2) is valid when

\[
\begin{align*}
\underline{u}_-(t) &= -\frac{1}{50} - \frac{2}{5}\sqrt{2501 + \frac{1}{5}\sqrt{25t^2 - 20t + 10001}}, \\
\underline{u}_+(t) &= \frac{1}{3} - \frac{2}{3}\sqrt{10 + \frac{1}{3}\sqrt{9t^2 + 120t + 4000}}.
\end{align*}
\]

Applying the iteration method described above one can show that the extremal solutions of (3.16) in \( \{ u \in Y \mid \underline{u}_- \leq u \leq \underline{u}_+ \} \) are

\[
\begin{align*}
u(t) &= \sqrt{t^2 + 400 - 20}, \\
u(t) &= \frac{1}{2} - 4\sqrt{34} + \sqrt{t^2 + 24t + 544}.
\end{align*}
\]

4. Existence Results for Second Order BVP’s

This section is devoted to the study of the phi-Laplacian boundary value problem (BVP)

\[
\begin{align*}
-\frac{d}{dt}(p(t)\phi(u'(t))) &= f(t, u, u') \text{ for a.e. } t \in J := (a, b), \\
\lim_{t \to a^+} p(t)\phi(u'(t)) &= c(u, u'), \\
\lim_{t \to b^-} u(t) &= d(u, u'),
\end{align*}
\]

(4.1)

where \( -\infty \leq a < b \leq \infty, \ p \in C(J), \ \phi : \mathbb{R} \to \mathbb{R}, \ f : J \times C(J) \times C(J) \to \mathbb{R} \) and \( c, d : C(J) \times C(J) \to \mathbb{R} \).

Denote

\[
Z := \{ h \in X \mid \int_r^b h(t) \, dt := \lim_{s \to b^-} \int_s^b h(t) \, dt \text{ is finite for an } r \in J \},
\]

(4.2)
where $X$ is defined by (3.3). Assuming that $Z$ is ordered a.e. pointwise, and that $C(J)$ is ordered pointwise, we shall show that the BVP (4.1) has extremal solutions in the pointwise ordered set $Y$ defined in (3.2), if the functions $p, \phi, f, c$ and $d$ satisfy the following hypotheses:

(φ) $\phi$ is an increasing homeomorphism from an open interval $I$ of $\mathbb{R}$ onto $\mathbb{R}$.

(φp) $p(t) > 0$ and $|\int_t^{b-} \phi^{-1}(\frac{K}{p(s)})ds| < \infty$, for all $t \in J$ and $K \in \mathbb{R}$.

(f0) $f(\cdot, u, u')$ is Lebesgue measurable and $h_- \leq f(\cdot, u, u') \leq h_+$, for all $u \in C^1(J)$ and for some $h_\pm \in Z$.

(f1) $f(\cdot, u, u') \leq f(\cdot, v, v')$ whenever $u, v \in C^1(J)$, $u \leq v$ and $u' \geq v'$.

(c1) $c_\pm \in \mathbb{R}$, and $c_- \leq c(v, v') \leq c(u, u') \leq c_+$ if $u, v \in C^1(J)$, $u \leq v$ and $u' \geq v'$.

(d1) $d_\pm \in \mathbb{R}$, and $d_- \leq d(u, u') \leq d(v, v') \leq d_+$ if $u, v \in C^1(J)$, $u \leq v$ and $u' \geq v'$.

The method is the same as in Section 3, that is, we shall first convert the BVP (4.1) to an integral equation, and then apply the fixed point result of Lemma 3.3.

**Lemma 4.1.** Assume that the hypotheses (φ) and (φp) hold. If $u \in Y$ and $f(\cdot, u, u) \in Z$, then $u$ is a solution of the BVP (4.1) if and only if

$$u(t) = d(u, u') - \int_t^{b-} \phi^{-1}\left(\frac{1}{p(s)}[c(u, u') - \int_s^a f(x, u, u') dx]\right)ds, \quad t \in J. \quad (4.3)$$

**Proof.** Assume that $u \in Y$ is a solution of (4.1), and that $f(\cdot, u, u') \in Y$. The differential equation of (4.1) and the definition (3.2) of $Y$ ensure that

$$\int_r^s f(t, u, u') dt = -\int_r^s \frac{d}{dt}(p(t)\phi(u'(t)))dt = p(r)\phi(u'(r)) - p(s)\phi(u'(s)),$$

when $a < r \leq s < b$. This result and the first boundary condition of (4.1) imply that

$$u'(s) = \phi^{-1}\left(\frac{1}{p(s)}[c(u, u') - \int_a^s f(x, u, u') dx]\right), \quad s \in J. \quad (4.4)$$
In view of the hypotheses \((\phi)\) and \((\phi p)\) we can integrate (4.4) and apply the second boundary condition of (4.1) to obtain (4.3).

The converse part of the proof is trivial. \(\square\)

The main existence result for the BVP (4.1) reads as follows.

**Theorem 4.1.** Assume that the hypotheses \((\phi)\), \((\phi p)\), \((f_0)\), \((f_1)\), \((c_1)\) and \((d_1)\) hold. Then the BVP (4.1) has least and greatest solutions in \(Y\), and they are increasing with respect to \(f\) and \(d\) and decreasing with respect to \(c\).

**Proof.** The given hypotheses imply that the equation

\[
Gu(t) := d(u, u') - \int_t^{b-} \phi^{-1} \left( \frac{1}{p(s)} \left[ c(u, u') - \int_{a+}^s f(x, u, u') \, dx \right] \right) \, ds,
\]

\(t \in J, \quad (4.5)\)

defines a mapping \(G : C^1(J) \to C^1(J)\). Define now a partial ordering in \(C^1(J)\) by

\[
u \leq_1 v \text{ iff } u \leq v \text{ and } u' \geq v'.
\]

(4.6)

The relations

\[
\begin{align*}
\underline{u}(t) & := d+ - \int_t^{b-} \phi^{-1} \left( \frac{1}{p(s)} \left[ c_+ - f_+^s h_-(x) \, dx \right] \right) \, ds, \quad t \in J, \\
\overline{u}(t) & := d_- - \int_t^{b-} \phi^{-1} \left( \frac{1}{p(s)} \left[ c_- - f_-^s h_+(x) \, dx \right] \right) \, ds, \quad t \in J,
\end{align*}
\]

(4.7)

define functions \(\underline{u}, \overline{u} \in Y\) for which \(\underline{u} \leq_1 \overline{u}\). Moreover, it is easy to see that \(\underline{u} \leq_1 Gu \leq_1 Gv \leq_1 \overline{u}\) for all \(u, v \in C^1(J), u \leq_1 v\). Thus \(G\) is increasing and its range is order-bounded.

Let \(W\) be a well-ordered chain in \(\text{ran } G\). By the definitions of \(\leq_1\) and \(G\) this implies that the set \(V = \{v' \mid v \in W\}\) is an inversely well-ordered chain in \(C(J)\), and \(\overline{v} \leq v' \leq \underline{v}'\) for each \(v' \in V\). In particular, \(\overline{v}'(t) \leq v'(t) \leq \underline{v}'(t)\) for all \(t \in J\) and \(v' \in V\). The given hypotheses ensure also that \(V\) is equicontinuous. Thus there exists by Lemma 3.2.b) a decreasing sequence \((v'_n)\) of \(V\) which converges locally uniformly in \(J\) to \(w = \inf V\).

The given hypotheses and the definitions of \(\leq_1\) and \(G\) imply also that \(W\) is a well-ordered and equicontinuous chain in \(C(J)\), and that \(\underline{u} \leq u \leq \overline{u}\) for each \(u \in W\). It then follows from Lemma 3.2.a) that \(u = \sup W\) exists in \(C(J)\), and there exists an increasing sequence \((u_n)\) in \(W\) which converges locally uniformly in \(J\) to \(u\). Denoting \(w_n = \max\{u_n, v_n\}, n \in \mathbb{N}\), where maximum is taken with respect to \(\leq_1\), then \(u_n \leq w_n \leq u\) in \(C(J)\) and \(v'_n \geq w'_n \geq w\) in \(C(J)\). These relations and the above proof imply that \(w_n \to u\) and \(w'_n \to w\) locally uniformly.
Thus \( w(t) = u'(t) \) in \( J \), by [2], Theorem 8.6.3, so that \( u \in C^1(J) \). This result and the definitions of \( u \) and \( w \) imply that \( u = \sup W \) in \( (C^1(J), \leq_1) \). Similarly one can show that each inversely well-ordered chain of \( \text{ran } G \) has an infimum.

The above proof shows that the operator \( G \) defined by (4.3) satisfies the hypotheses of Lemma 3.3, whence \( G \) has a least fixed point \( u^* \) and a greatest fixed point \( u^* \). According to Lemma 4.1 \( u^* \) and \( u^* \) are least and greatest solutions of the BVP (4.1). The last assertion is an easy consequence of the last assertion of Lemma 3.3 and the definition (4.5) of \( G \).

If we replace the hypothesis \((f_0)\) by the following hypothesis:

\[(f_2)\quad f(\cdot, u, u') \in Z \text{ for all } u \in C^1(J), \text{ and there exist } \underline{u}, \underline{u} \in C^1(J) \text{ such that } \underline{u} \leq \overline{u}, \underline{u} \leq_1 Gu \text{ and } G\overline{u} \leq_1 \overline{u}, \text{ where } G \text{ is defined by (4.5)},\]

we get the following result.

**Proposition 4.1.** Assume that the hypotheses \((\phi), (\phi p), (f_1), (f_2), (c_1)\) and \((d_1)\) hold. Then the BVP (4.1) has a least and a greatest solution in \( \{u \in Y \mid u \leq u \leq \overline{u}\} \).

As a special case we obtain an existence result for the BVP

\[
\left\{ \begin{array}{l}
-\frac{d}{dt}(p(t)\phi(u'(t))) = g(t, u(t), u'(t)) \text{ for a.e. } t \in J, \\
\lim_{t \to a^+} p(t)\phi(u'(t)) = c, \quad \lim_{t \to b^-} u(t) = d.
\end{array} \right.
\] (4.8)

**Corollary 4.1.** Let the hypotheses \((\phi)\) and \((\phi p)\) hold, and let \( g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfy the following hypotheses:

\[(g_0)\quad g(\cdot, u(\cdot), u'(\cdot)) \text{ is Lebesgue measurable and } h_- \leq g(\cdot, u(\cdot), u'(\cdot)) \leq h_+ \text{ for all } u \in C^1(J) \text{ and for some } h_\pm \in \mathbb{R}.
\]

\[(g_1)\quad g(t, x, z) \leq g(t, y, w) \text{ for a.e. } t \in J \text{ and whenever } x \leq y \text{ and } z \geq w \text{ in } \mathbb{R}.
\]

Then the BVP (4.8) has for each choice of \( c, d \in \mathbb{R} \) least and greatest solutions in \( Y \). Moreover, these solutions are increasing with respect to \( g \) and \( d \) and decreasing with respect to \( c \).

**Proof.** If \( c, d \in \mathbb{R} \), the BVP (4.8) is reduced to (4.1) when we define

\[
\left\{ \begin{array}{l}
f(t, u, u') = g(t, u(t), u'(t)), \quad t \in J, \; u \in C^1(J), \\
c(v, v') \equiv c, \; v \in C^1(J), \quad d(v, v') \equiv d, \; v \in C^1(J).
\end{array} \right.
\] (4.9)
defined by (4.9), and hence also (4.8), has by Theorem 4.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 4.1. 

In the case when \( \phi \) is defined by
\[
\phi(x) = \frac{x}{\sqrt{1-x^2}}, \quad x \in (-1, 1),
\]
the operator \( G \) given by (4.5) can be rewritten as
\[
Gu(t) = d(u, u') - \int_t^b \frac{c(u, u') - \int_{a_1}^s f(x, u, u') \, dx}{\sqrt{p^2(s) + (c(u, u') - \int_{a_1}^s f(x, u, u') \, dx)^2}} \, ds,
\]
t \in J. \quad (4.10)

This formula shows that \( d(u, u') - (b - t) \leq Gu(t) \leq d(u, u') + (b - t) \) whenever the right hand side of (4.10) is defined. Thus we have the following result.

**Corollary 4.2.** If \(-\infty \leq a < b < \infty\), the IVP
\[
\begin{cases}
\frac{d}{dt}\left(\frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}}\right) = f(t, u, u') \quad \text{a.e. in } J = (a, b), \\
\lim_{t \to a^+} \frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}} = c(u, u'), \quad u(b) = d(u, u'),
\end{cases}
\]
has least and greatest solutions if \( p(t) > 0 \) for all \( t \in J \), if the hypothesis \((d_1)\) holds, and if \( f(\cdot, u, u') \leq f(\cdot, v, v') \) in \( Z \) and \( c(v, v') \leq c(u, u') \) in \( R \) whenever \( u \leq v \) and \( u' \geq v' \) in \( C(J) \).

**Example 4.1.** Consider the singular BVP
\[
\begin{cases}
-\frac{d}{dt}\left(\frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}}\right) = h(t) + K[q(t) \int_1^2 (u(s) - u'(s)) \, ds] \quad \text{a.e. in } J, \\
\lim_{t \to a^+} \frac{p(t)u'(t)}{\sqrt{1-u'(t)^2}} = c \cdot \frac{u'(t)}{\sqrt{1-u'(t)^2}}, \\
u(3) = \left[10^{|u(1) - u'(1)|}\right] / [1 + |10^{|u(1) - u'(1)|}|],
\end{cases}
\]
where \( J = (0, 3) \), \( p \in C(J) \), \( \lim_{t \to a^+} p(t) = 0 \), \( p(t) > 0 \) for \( t \in J \), \( q \in L_1^1(J) \), \( h \in Z \), \( K, c \geq 0 \), and \([z]\) denotes the greatest integer \( \leq z \). The problem (4.12) is of the form (4.11) with
\[
\begin{cases}
f(t, u, u') = h(t) + K[q(t) \int_1^2 (u(s) - u'(s)) \, ds], \quad t \in J, \\
c(u, u') = c \cdot \frac{u'(t)}{\sqrt{1-u'(t)^2}} \quad \text{and} \quad d(u, u') = \left[10^{|u(1) - u'(1)|}\right] / [1 + |10^{|u(1) - u'(1)|}|].
\end{cases}
\]

It is easy to see that the hypotheses of Corollary 4.2 hold, whence (4.12) has least and greatest solutions.
Remark 4.1. If \( h(t) = \frac{1}{2} \sin \frac{1}{t}, \ t \in J, \) then \( h \) and the function \( f(\cdot, u, u') \) defined in (4.13) belong to \( Z, \) but not to \( L^1(J). \)

Example 4.2. Determine least and greatest solutions of the BVP

\[
\begin{aligned}
- \frac{d}{dt} \left( \frac{tu'(t)}{\sqrt{1-u'(t)^2}} \right) &= t - 1 + \frac{1}{10} \left[ 10 \int_1^2 (u(s) - u'(s)) ds \right] \text{ a.e. in } (0, 3), \\
\lim_{t \to 0^+} \frac{tu'(t)}{\sqrt{1-u'(t)^2}} &= 0, \\
u(3) &= \frac{10}{1+10|u(1)-u'(1)|}.
\end{aligned}
\tag{4.14}
\]

Solution. (4.14) is a special case of (4.12) when \( p(t) = t, \ h(t) = t - 1, \ K = \frac{1}{10}, \ q(t) \equiv 10 \) and \( c = 0. \) Thus (4.14) has extremal solutions. The functions \( \underline{u} \) and \( \overline{u} \) defined by (4.7) can be calculated, and one obtains

\[
\underline{u}(t) = -1 + \sqrt{365} - \sqrt{t^2 - 44t + 488}, \quad \overline{u}(t) = 1 + \sqrt{445} - \sqrt{t^2 + 36t + 328}.
\]

Calculating the iterations \( G^n \underline{u}, \) where \( G \) is defined by (4.10), it turns out that \( G^2 \underline{u} = G^3 \underline{u}. \) Thus \( u_* = G^2 \underline{u} \) is a least solution of (4.14) by [4], Corollary 1.1.2. Similarly, one can show that \( u^* = G^2 \overline{u} = G^3 \overline{u} \) is a greatest solution of (4.14). The exact expressions of these solutions are

\[
\begin{aligned}
\underline{u}_*(t) &= -\frac{39}{38} + \frac{1}{5} \sqrt{941} - \frac{1}{5} \sqrt{25t^2 - 440t + 2036}, \\
u^*(t) &= \frac{38}{39} + \frac{2}{5} \sqrt{386} - \frac{1}{5} \sqrt{25t^2 + 230t + 629}.
\end{aligned}
\tag{4.15}
\]

Remarks 4.3. As for uniqueness results for phi-Laplacian initial and boundary value problems see, e.g., [1, 5]. Problems of the form (2.1), (3.1) and (4.1) include many kinds of special types. For instance, they can be

– singular, because a case \( \lim_{t \to a^+} p(t) = 0 \) is allowed;
– functional, because the functions \( c, \ d \) and \( f \) may depend functionally on \( u \) and/or \( u' \);
– discontinuous, because \( c, \ d \) and \( f \) can depend discontinuously on \( u \) and/or \( u' \);
– problems on unbounded domains, because cases \( a = -\infty \) and/or \( b = \infty \) are included;
– \( p \)-Laplacian when \( \phi \) is defined by (3.11).
References


