ON THE SOLUTION OF PARAMETRIC LINEAR COMPLEMENTARITY PROBLEMS

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Abstract: This paper presents the solution procedure of a linear complementarity problem where the associated matrix as well as the column vector are both parametric with a special case that the matrix remains a $P$-matrix and the solution basis remains the same for any input of the parameters. We have also introduced another algorithm applicable for $P$-matrices for getting the initial solution of the above problem. Illustrated numerical example proves the efficiency of the proposed algorithms.

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1. Introduction

Linear Complementarity Problem, denoted by LCP($q, M$), is a general problem
of finding nonnegative solution \((w, z)\) to a system of equations of the following form:

\[
\begin{align*}
w - Mz &= q, \\
w &\geq 0, \quad z \geq 0, \\
w^Tz &= 0,
\end{align*}
\]

where \(M\) is square matrix of order \(n\) and \(w, z, q\) are \(n\)-dimensional column vectors.

LCP has wide range of application in optimization, game theory, economic equilibrium theory, mechanics, approximate solutions to differential equations, classification of square matrices, etc.

The popularity of the LCP in mathematical programming has led to a variety of algorithms \([1, 3, 5, 8, 10, 11, 12]\) attempting to solve the problem. Among these variety of algorithms, the most renowned algorithm for solving the above LCP is due to Lemke \([8]\), called the complementary pivot algorithm. This algorithm gives unique solution when \(M\) is a \(P\)-matrix. The principal pivoting algorithm given by Cottle and Dantzig \([3]\) also gives unique solution when \(M\) is a \(P\)-matrix.

A LCP is known as a parametric LCP if \(M\) and \(q\) are parametric in nature. A parametric LCP can be defined as: find vectors \((w, z)\) satisfying

\[
\begin{align*}
\begin{align*}
w - M(\alpha)z &= q(\lambda) = q + \lambda q^\star, \\
w &\geq 0, \quad z \geq 0, \\
w^Tz &= 0,
\end{align*}
\end{align*}
\]

where \(M(\alpha)\) is a parametric matrix of order \(n\), and \(q(\lambda)\) is a parametric \(n\)-dimensional column vector. Here \(\alpha\) and \(\lambda\) are two parameters. The solution as well as its basis are affected by the input \(M(\alpha)\) and \(q(\lambda)\).

Cottle \([2, 4]\) first studied the parametric LCP. After that authors like Kaniko \([6, 7]\), Murty \([9]\), Xiao \([13]\), Xiao and Harkar \([14]\), etc. have contributed in the field of parametric LCP. Xiao \([13]\) considered the case when \(M\) is parametric. The algorithm developed by Murty \([9]\) is applicable when only \(q\) is parametric. It is most useful when \(M\) is a \(P\)-matrix. It solves the LCP \((q(\lambda), M)\) for some fixed value of \(\lambda\) by any method such as complementarity pivot method \([8]\) or the principal pivoting method \([3]\), and then obtain the range of \(\lambda\) by using some condition for which the complementary basic vector remains feasible.

In this paper we develop an algorithm for solving parametric LCP associated with \(P\)-matrix when both \(M\) and \(q\) are parametric. We determine the ranges of \(\alpha\) and \(\lambda\) such that the basic variables remain fixed for any disturbances of
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$M(\alpha)$ and $q(\lambda)$ within the range. The algorithm proposed by us reveals the superiority over the complementarity pivot and principal pivot method both with regard to the number of iterations and computational time.

The organization of the paper is as follows: following the introduction, we present the notations used in this paper. An Algorithm to solve LCP when $M$ is a $P$-matrix is presented in Section 3. Section 4 presents another algorithm to solve the LCP where both $M$ and $q$ are parametric. Conclusions are presented in Section 5.

2. Notations

Let the $i$-th row and $j$-th column of $M(\alpha)$ be denoted by $M_i(\alpha)$ and $M_j(\alpha)$, respectively. Let $I \subset \{1, 2, \ldots, n\}$ and $J \subset \{1, 2, \ldots, n\}$. Then $M_{IJ}(\alpha)$ is the sub-matrix of $M(\alpha)$ whose rows and columns are indexed by $I$ and $J$, respectively.

Let $B \subset \{1, 2, \ldots, n\}$ be the indexed set and $N$ be its complement such that $z_B > 0$, $w_N \geq 0$, $z_N = w_B = 0$.

3. An Algorithm to Solve LCP, when $M$ is a $P$-matrix

Step 1. Start from the first row and see if $q_1 \geq 0$ or $q_1 < 0$. If $q_1 \geq 0$ then take $M_1z^T = 0$. If $q_1 < 0$ then take $M_1z^T = -q_1$. Repeat the same process for the rest of the rows.

Step 2. Thus we have a system of $n$-linear equation in $n$-unknowns. By Gauss elimination method we reduce the above matrix $Mz = q$ to an upper triangular matrix $Uz = b$. Now these system can be solved by back substitution. Hence the solution is

$$z_k = \begin{cases} \frac{b_k - \sum_{j=k+1}^{n} a_{kj}z_j}{a_{kk}}, & \text{if } \left( \frac{b_k - \sum_{j=k+1}^{n} a_{kj}z_j}{a_{kk}} \right) > 0, \\ 0, & \text{if } \left( \frac{b_k - \sum_{j=k+1}^{n} a_{kj}z_j}{a_{kk}} \right) \leq 0. \end{cases}$$

Step 3. Now corresponding to the value of $z_k = 0$, $(1 \leq k \leq n)$ we find the value of $w_k$ from the $k$-th equation of the original system of equation (1).
3.1. Numerical Example

Let
\[ M = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -2 \\ -1 & 3 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix}. \]

The original problem is
\[ \begin{align*}
w_1 - z_1 - z_2 + z_3 &= -4, \\
w_2 + z_1 - 2z_2 + 2z_3 &= -5, \\
w_3 + z_1 - 3z_2 - 2z_3 &= 3, \\
w &\geq 0, \quad z &\geq 0, \\
w^Tz &= 0.
\end{align*} \quad (3) \]

Step 1. \( q_1 < 0, \) then \( z_1 + z_2 - z_3 = 4, \) \( q_2 < 0, \) then \( -z_1 + 2z_2 - 2z_3 = 5, \) \( q_3 \geq 0, \) then \( z_1 - 3z_2 - 2z_3 = 0. \)

Step 2. Eliminate \( z_1 \) from the second and third equations, which yields
\[ \begin{align*}
z_1 + z_2 - z_3 &= 4, \\
z_2 - z_3 &= 3, \\
-4z_2 - z_3 &= -4.
\end{align*} \]

Again we eliminate \( z_2 \) from the third equation and have the final triangular system in the form
\[ \begin{align*}
z_1 + z_2 - z_3 &= 4, \\
z_2 - z_3 &= 3, \\
-5z_3 &= 8.
\end{align*} \]

By back substitution we get
\[ z_3 = 0 \text{ as } -8/5 < 0, \quad z_2 = 3, \quad z_1 = 1. \]

From the original problem we get the value of \( w_3 \) from the third equation as
\[ w_3 + 1 + 9 = 3 \Rightarrow w_3 = 11. \]

Hence the solution is
\[ w_1 = 0, \ w_2 = 0, \ w_3 = 11, \ z_1 = 1, \ z_2 = 3, \ z_3 = 0. \]
4. Algorithm to Solve the LCP where Both $M$ and $q$
are Parametric

Step 1. Initialize $\lambda$. Choose $\alpha_i = \alpha_0 \ \forall i$ and find the solution of the LCP using Algorithm 2. Check the following three conditions.

(a) $\det M_BB(\alpha) > 0$.
(b) $\det M_BB / j(\alpha), \ \forall j \in B$ (whose $j$-th column is replaced by $-q_B$).
(c) $\det L_j \geq 0$, where

\[ L_j = \begin{pmatrix} M_{BB}(\alpha) & q_B \\ M_j(\alpha) & q_j \end{pmatrix}, \ \forall j \in N. \]

If the above three conditions hold for $\alpha_i = \alpha_0, \ \forall i$, then goto Step 2.

Step 2. With the same solution basis for any perturbation of the parameters find the range of $\alpha_i, \ \forall i$ from condition (b).

i.e. $\{ y = \lceil x \rceil | \max_{\alpha_i > x}(x) \} \leq \alpha_i \leq \{ y = \lfloor x \rfloor | \min_{\alpha_i < x}(x) \}$.

Step 3. In this range if condition (a) and (c) are satisfied then the basic variables stays the same for any perturbations of the parameters within the range. Goto Step 4. Otherwise with some other values of $\alpha_i, \ \forall i$, goto Step 1.

Step 4. The perturbed solution with this basic variables are given by

\[ z_B = \frac{\det M_{BB} / j(\alpha)}{\det M_{BB}(\alpha)} + \lambda \frac{\det M_{BB} / k(\alpha)}{\det M_{BB}(\alpha)} , \ \forall j, k \in B , \]

i.e.,

\[ z_B = \overline{q}_B + \lambda \overline{q}^*_B. \]

Here the $j$-th column is replaced by $-q_B$ and the $k$-th column is replaced by $-q^*$.

\[ w_N = \frac{\det L_j}{\det M_{BB}(\alpha)} + \lambda \frac{\det L_k}{\det M_{BB}(\alpha)} , \ \forall j, k \in N , \]

i.e.,

\[ w_N = \overline{q}_N + \lambda \overline{q}^*_N , \]

where

\[ L_k = \begin{pmatrix} M_{BB}(\alpha) & \overline{q}_B \\ M_k(\alpha) & \overline{q}_k \end{pmatrix}, \ \forall k \in N , \]

\[ z_N = w_B = 0. \]
Therefore
\[ \vec{q} = \left( \frac{\vec{q}_B}{\vec{q}_N} \right), \quad \vec{q}^* = \left( \frac{\vec{q}_B^*}{\vec{q}_N^*} \right). \]

With \( l = 1 \), goto Step 5.

**Step 5.** For different values of \( \alpha_i (\forall i) \) within this range, find \( \vec{q} \) and \( \vec{q}^* \) and determine the range of \( \lambda \) so that the complementarity basic vectors remains feasible.

**Step 6.** Compute \( \lambda^l \), \( \bar{\lambda} \), the lower and upper characteristic values associated with the complementary basic vector from the following:

\[ \lambda^l = \begin{cases} -\infty, & \text{if } \vec{q}_i^* \leq 0, \ \forall i, \\ \max \left\{ -\frac{\vec{q}_i}{\vec{q}_i^*} : i \text{ such that } \vec{q}_i^* > 0 \right\}, & \text{otherwise}. \]

\[ \bar{\lambda} = \begin{cases} \infty, & \text{if } \vec{q}_i^* \geq 0, \ \forall i, \\ \min \left\{ -\frac{\vec{q}_i}{\vec{q}_i^*} : i \text{ such that } \vec{q}_i^* < 0 \right\}, & \text{otherwise}. \]

The complementary basic vector is feasible in the closed interval \( \lambda^l \leq \lambda \leq \bar{\lambda} \) and hence the solution is

\[ z_B = \vec{q}_B + \lambda \vec{q}_B^*, \quad w_N = \vec{q}_N + \lambda \vec{q}_N^*, \quad z_N = w_B = 0, \]

\( l = l + 1 \) and goto Step 5.

The solution for different values of \( \alpha \) will remain feasible within the range of \( \lambda \) given by

\[ \max \{ \lambda^l, \ \forall l \} \leq \lambda \leq \min \{ \bar{\lambda}, \ \forall l \}. \]

### 4.1. Numerical Example

\[ M(\alpha) = \begin{pmatrix} 1 & 1 & -1 \\ -1 + 2\alpha_1 & 2 + \alpha_2 & -2 + \alpha_3 \\ -1 & 3 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix}, \]

\[ q^* = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad q(\lambda) = q + \lambda q^*. \]

**Step 1.** Taking \( \lambda = 0 \) and \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \). We get the solution using algorithm 2 as

\[ z_1 = 1, \quad z_2 = 3, \quad w_3 = 11, \quad z_3 = w_1 = w_2 = 0. \]
Here $B = \{1, 2\}$, $N = \{3\}$. For $\alpha_1 = \alpha_2 = \alpha_3 = 0$:

(a) $\det M_{BB}(\alpha) = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3 > 0$.

(b) $\det M_{BB/1}(\alpha) = \begin{vmatrix} 4 & 1 \\ 5 & 2 + \alpha_2 \end{vmatrix} = 3 + 4\alpha_2 > 0$, $\Rightarrow \alpha_2 > -3/4 = -0.75$,

$$\det M_{BB/2}(\alpha) = \begin{vmatrix} 1 & 4 \\ -1 + 2\alpha_2 & 5 \end{vmatrix} = 9 - 8\alpha_1 > 0, \Rightarrow \alpha_1 < 9/8 = 1.1,$$

$$\{y = [-.75] | \max_{\alpha_i > x} (\cdot) \leq \alpha_i \leq \min_{\alpha_i < x} \{y = [1.1] | \min_{\alpha_i < x} (\cdot) \}.$$

Step 3. Now for $0 \leq \alpha_i \leq 1$, $\forall i$:

(a) $\det M_{BB}(\alpha) = \begin{vmatrix} 1 & 1 \\ -1 + 2\alpha_1 & 2 + \alpha_2 \end{vmatrix} = 3 + \alpha_2 - 2\alpha_1 > 0$.

(c) $\det L_j = \begin{vmatrix} 1 & 1 & -4 \\ -1 + 2\alpha_1 & 2 + \alpha_2 & -5 \\ -1 & 3 & 3 \end{vmatrix} = 33 - 30\alpha_1 - \alpha_2 > 0$.

Hence for any perturbation within this range the basic variables stay the same.

Step 4. The perturbed solution is

$$z_1 = \frac{3 + 4\alpha_2}{3 - 2\alpha_1 + \alpha_2} + \lambda \frac{1 + \alpha_2}{3 - 2\alpha_1 + \alpha_2} = \frac{3 + 4\alpha_2}{3 - 2\alpha_1 + \alpha_2} \text{ (as $\lambda = 0$)},$$

$$z_2 = \frac{9 - 8\alpha_1}{3 - 2\alpha_1 + \alpha_2} + \lambda \frac{2 - 2\alpha_2}{3 - 2\alpha_1 + \alpha_2} = \frac{9 - 8\alpha_1}{3 - 2\alpha_1 + \alpha_2} \text{ (as $\lambda = 0$)},$$

$$w_3 = \frac{-30\alpha_1 - \alpha_2 + 33}{3 - 2\alpha_1 + \alpha_2} + \lambda \frac{2 - 4\alpha_1 - 2\alpha_2}{3 - 2\alpha_1 + \alpha_2} = \frac{-30\alpha_1 - \alpha_2 + 33}{3 - 2\alpha_1 + \alpha_2} \text{ (as $\lambda = 0$)},$$

$$z_3 = w_1 = w_2 = 0.$$
Step 5. When $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\bar{q} = \begin{pmatrix} 1 \\ 3 \\ 11 \end{pmatrix}, \quad \bar{q}^r = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix},$$

$\Lambda^1 = -3$, $\bar{\Lambda}^1 = \infty$, i.e., $-3 \leq \lambda \leq \infty$. The solution is

$$z_1 = 1 + \lambda/3, \quad z_2 = 3 + 2\lambda/3, \quad w_3 = 11 + 2\lambda/3, \quad z_3 = w_1 = w_2 = 0.$$ 

When $\alpha_1 = 0$, $\alpha_2 = 1$

$$\bar{q} = \begin{pmatrix} 7/4 \\ 9/4 \\ 8 \end{pmatrix}, \quad \bar{q}^r = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix},$$

$\Lambda^2 = -7/2$, $\bar{\Lambda}^2 = \infty$, i.e., $-7/2 \leq \lambda \leq \infty$. The solution is

$$z_1 = 7/4 + \lambda/2, \quad z_2 = 9/4, \quad w_3 = 8, \quad z_3 = w_1 = w_2 = 0.$$ 

When $\alpha_1 = 1/2$, $\alpha_2 = 1/2$

$$\bar{q} = \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}, \quad \bar{q}^r = \begin{pmatrix} 3/5 \\ 2/5 \\ -2/5 \end{pmatrix},$$

$\Lambda^3 = -10/3$, $\bar{\Lambda}^3 = 35/2$, i.e., $-10/3 \leq \lambda \leq 35/2$. The solution is

$$z_1 = 2 + 3\lambda/5, \quad z_2 = 2 + 2\lambda/5, \quad w_3 = 7 - 2\lambda/5, \quad z_3 = w_1 = w_2 = 0.$$ 

When $\alpha_1 = 1$, $\alpha_2 = 0$

$$\bar{q} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \quad \bar{q}^r = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

$\Lambda^4 = -1/2$, $\bar{\Lambda}^4 = 3/2$, i.e., $-1/2 \leq \lambda \leq 3/2$. The solution is

$$z_1 = 3 + \lambda, \quad z_2 = 1 + 2\lambda, \quad w_3 = 3 - 2\lambda, \quad z_3 = w_1 = w_2 = 0.$$ 

For these four values of $\alpha_i$’s the solution will remain feasible within the range $-1/2 \leq \lambda \leq 3/2$. 
5. Conclusions

We have developed two algorithms. The first algorithm though is applicable only for $P$-matrices, it takes less time and less iterations than any other algorithms developed for solving LCP associated with $P$-matrices. The second algorithm is applicable for the $LCP(M(\alpha), q(\lambda))$ only when the first algorithm can finds a unique solution for any parametric input of $M$ and $q$.

References


