ON PRIME SUBMODULES OF MULTIPLICATION MODULES

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Abstract: The main aim of this paper is extending Anderson’s Theorem [2] to multiplication modules and extending Cohen’s Theorem to multiplication modules without assumption that $M$ is finitely generated.

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1. Introduction

Throughout this paper $R$ is a commutative ring with identity and $M$ is a unitary $R$-module. A proper submodule $P$ of $M$ is called prime if for any $r \in R$ and $m \in M$ with $rm \in P$ we have $m \in P$ or $rM \subseteq P$. An $R$-module $M$ is called a multiplication module provided that for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. The main aim in the first section of
this paper is extending Anderson’s Theorem [2] to multiplication modules. We show that if \( M \) is a multiplication module and every minimal prime submodule of \( M \) is finitely generated then \( M \) is finitely generated. This shows that the extension of Cohen’s Theorem for a multiplication module \( M \) is true without the assumption that \( M \) is finitely generated. In the second section, we shall give some results concerning the multiplication modules and the Jacobson radical of modules.

**Definition 1.1.** Let \( N \) be a submodule of \( M \) and \( N = IM, I \triangleleft R \), then we say that \( I \) is an ideal of \( N \).

**Definition 1.2.** Let \( N_1 = I_1M \) and \( N_2 = I_2M \). Then we define the product of \( N_1 \) and \( N_2 \) by \( N_1N_2 = I_1I_2M \).

**Lemma 1.3.** The product of two submodules in a multiplication module is well-defined and it is a submodule of \( M \).

**Proof.** Let \( N_1 = I_1M = I_2M = N_2 \) and \( K_1 = J_1M = J_2M = K_2 \). Now we have

\[
(I_1J_1)M = I_1(J_1M) = I_1(J_2M) = J_2(I_1M) = J_2(I_2M) = (I_2J_2)M
\]

therefore the product is well-defined. \( \square \)

**Remark.** If \( M \) is a multiplication module and \( N = IM \), then \( N = IM = (N : M)M \).

**Lemma 1.4.** Let \( P \) be a prime submodule of \( R \)-module \( M \). Then \( (P : M) \) is a prime ideal of \( R \).

**Proof.** Evident. \( \square \)

**Lemma 1.5.** Let \( M \) be a multiplication \( R \)-module and \( N_1 = IM, N_2 = JM \) and \( I \subseteq J \). Then \( N_1 \subseteq N_2 \). Also if \( N_1 \subset N_2 \) then \( (N_1 : M) \subset (N_2 : M) \).

**Proof.** Evident. \( \square \)

**Theorem 1.6.** Let \( P \) be a prime submodule of a multiplication \( R \)-module \( M \), and \( UV \subseteq P \) for submodules \( U \) and \( V \) of \( M \) then \( U \subseteq P \) or \( V \subseteq P \).

**Proof.** Let \( P \) be prime, \( U = I_1M \) and \( V = I_2M \). Now if \( UV = I_1I_2M \subseteq P \). Then \( I_1I_2 \subseteq (P : M) \) so that either \( I_1 \subseteq (P : M) \) or \( I_2 \subseteq (P : M) \) therefore \( I_1M \subseteq (P : M)M \) or \( I_2M \subseteq (P : M)M \). Hence \( U \subseteq P \) or \( V \subseteq P \). \( \square \)

**Definition 1.7.** Let \( M \) be a multiplication module and \( N \) be a submodule of \( M \). Then \( N \) is called nilpotent if \( N^k = 0 \) for some positive integer \( k \), where \( N^k \) denotes the product of \( N \), \( k \) times.
**Theorem 1.8.** Let $M$ be a multiplication module. A submodule $N$ of $M$ is nilpotent if and only if for every $I$, the ideal of $N$, $I^k \subseteq \text{ann}(M)$, for some positive integer $k$.

**Proof.** Let $I$ be the ideal of $N$ then

$$N^k = 0 \iff I^k M = 0 \iff I^k \subseteq \text{ann}(M)$$

**Corollary 1.9.** Let $M$ be a faithful (torsion free) multiplication module and $N$ a submodule of $M$. Then $N$ is nilpotent if and only if every ideal of $N$ is nilpotent.

**Lemma 1.10.** Let $M$ be a multiplication module, $N_1$ and $N_2$ finitely generated submodules of $M$ then $N_1 N_2$ is also finitely generated.

**Proof.** Let $N_1 = I_1 M = \sum_{i=1}^{n} R a_i m_i$ and $N_2 = I_2 M = \sum_{j=1}^{k} R b_j m'_j$, where $m_i, m'_j \in M$ and $a_i \in I_1, b_j \in I_2$ for all $i, j$. We claim that

$$N_1 N_2 = \sum_{i,j} R a_i b_j m_i = \sum_{i,j} R a_i b_j m'_j \quad i = 1, \ldots, n, \quad j = 1, \ldots, k.$$ 

Clearly $\sum_{i,j} R a_i b_j m_i \subseteq N_1 N_2$. Now each element of $N_1 N_2$ is a finite sum of elements of the form $abm$, where $m \in M$, $a \in I_1$, $b \in I_2$. But we have $abm = a(\sum_{j=1}^{k} c_j b_j m'_j)$, where $c_j \in R$. Hence

$$abm = \sum_{j=1}^{k} c_j b_j (am'_j) = \sum_{j=1}^{k} c_j b_j (\sum_{i=1}^{n} d_{ij} a_i m_i),$$

where $d_{ij} \in R$. This shows that $abm \in \sum_{i,j} R a_i b_j m_i$. Similarly we have $N_1 N_2 = \sum_{i,j} R a_i b_j m'_j$. □

Anderson in [2] proved the following theorem: Let $R$ be a commutative ring with identity, and let $I \neq R$ be an ideal of $R$. If every prime ideal minimal over $I$ is finitely generated, then there are only finitely many prime ideals minimal over $I$.

Now we shall extend this result to multiplication modules.

**Theorem 1.11.** Let $M$ be a multiplication module and $N$ a proper submodule of $M$ such that every prime submodule of $M$ minimal over $N$ is finitely generated. Then there are only finitely many prime submodules of $M$ minimal over $N$. 
Proof. Let
\[ S = \{P_1P_2\ldots P_n : \text{each } P_i \text{ is a prime submodule minimal over } N\} \].

If for some \( C = P_1P_2\ldots P_n \in S \) we have \( C \subseteq N \), then by Theorem 1.6 any prime submodule \( P \) minimal over \( N \) contains some \( P_i \) so \( \{P_1, P_2, \ldots, P_n\} \) is the set of minimal prime submodules of \( N \). Therefore we may assume that \( N \) contain no element of \( S \). Now consider the set
\[ T = \{K|N \subseteq K \text{ and } C \not\subseteq K \text{ for each } C \in S\} \].

Now by Lemma 1.10 each element of \( S \) is finitely generated and therefore by Zorn’s lemma \( T \) has a maximal element \( P \). Now we shall prove that \( P \) is prime. Let \( am \in P \) with \( m \not\in P \) and \( a \not\in (P : M) \). So there exists, \( m_1 \in M \) such that \( am_1 \not\in P \). Hence \( P \subseteq P + Rm \) and \( P \subseteq P + Ram_1 \). Now by the maximality of \( P \) there exist elements \( P_1P_2\ldots P_n \) and \( P'_1P'_2\ldots P'_k \) in \( S \) such that \( P_1P_2\ldots P_n \subseteq P + Rm \) and \( P'_1P'_2\ldots P'_k \subseteq P + Ram_1 \). Now let \( b \in P_1P_2\ldots P_nP'_1P'_2\ldots P'_k \). Then there exist \( x \in (P_1 : M)(P_2 : M)\ldots(P_n : M) \) and \( y \in (P'_1 : M)(P'_2 : M)\ldots(P'_k : M) \) and \( m_2 \in M \) such that
\[ b = xym_2 = x(p' + ram_1) = xp' + raxm_1 = xp' + ra(p'' + tm) = xp' + rap'' + rtm \in P \],

where \( p', p'' \in P \) and \( r, t \in R \). This shows that \( P_1P_2\ldots P_nP'_1P'_2\ldots P'_k \subseteq P \), a contradiction, therefore \( P \) is a prime submodule. But then since \( P \supseteq N \), \( P \) contains a prime submodule \( Q \) minimal over \( N \) thus \( Q \in S \), a contradiction. \( \square \)

Now by [16] Theorem 3.7 and the above theorem we have the following result.

**Corollary 1.12.** Let \( R \) be commutative ring and \( M \) a multiplication module such that every minimal prime submodule of \( M \) is finitely generated, then \( M \) is finitely generated.

As a corollary we can prove the following theorem, which is due to Karakas [7].

**Corollary 1.13.** Let \( R \) be commutative ring and \( M \) a multiplication module. Then \( M \) is Noetherian if and only if, every prime submodule of \( M \) is finitely generated.
2. Some Results on Jacobson Module and Multiplication Module

**Definition 2.1.** Let $M$ be an $R$-module. The Jacobson radical of $M$ (denoted $J(M)$) is the intersection of all maximal submodules of $M$. If no maximal submodule exists then we set $J(M) = M$.

Similarly the Jacobson radical of $R$ will be denoted by $J(R)$.

**Remark.** We always have $J(R)M \subseteq J(M)$ where $R$ is a commutative ring and $M$ is an $R$-module.

MacCasland proved (see [10], Theorem 1.14) that if $R$ is a local ring and $M$ finitely generated $R$-module then $J(M) = J(R)M$. Now we shall show that the theorem is true for a multiplication $R$-module $M$ without the assumption that $M$ is finitely generated.

**Lemma 2.2.** Let $M$ be a non zero multiplication $R$-module then every proper submodule of $M$ is contained in a maximal submodule of $M$.

**Proof.** See [16], Theorem 2.5. \qed

**Lemma 2.3.** Let $R$ be a local ring and $M$ a multiplication $R$-module. Then $J(M) = J(R)M$.

**Proof.** By assumption $J(M) = IM$ for some ideal $I$. Since $I \subseteq J(R)$ so that $J(M) = IM \subseteq J(R)M$. But we always have $J(R)M \subseteq J(M)$, consequently $J(M) = J(R)M$. \qed

**Definition 2.4.** Let $M$ be a finitely generated $R$-module. We say that $M$ is of rank $n$ ($n \in \mathbb{Z}^+$) if $M$ has a minimal generating set of $n$ elements and does not have any minimal generating set of fewer than $n$ elements.

**Definition 2.5.** A submodule $N$ of $M$ is called small in $M$ if for any submodule $K$ of $M$, $N + K = M \implies K = M$.

**Lemma 2.6.** Let $M$ be multiplication $R$-module, $N \subseteq M$. Then the following conditions are equivalent:

(i) $N \subseteq J(M)$.

(ii) $N$ is small in $M$.

**Proof.** (i)$\implies$(ii) Suppose for some $C \leq M$, we have $M = N + C$. If $C \neq M$ then by Lemma 2.2, there exists a maximal submodule $B < M$ such that $C \subseteq B$.

But $N \subseteq J(M) \subseteq B$ implies that $N + C \subseteq B \neq M$. Thus $C = M$.

(ii)$\implies$(i) Suppose that for every $C \leq M$ such that $M = N + C$, we have $C = M$. Suppose then that $N \not\subseteq J(M)$. Then there is a maximal submodule $B < M$ such that $N \not\subseteq B$. Thus $N + B = M$, but $B \neq M$. This contradicts the hypothesis. Hence $N \subseteq J(M)$. \qed
**Theorem 2.7.** Let $R$ be a local ring, $M$ a multiplication $R$-module. Then every submodule of $M$ is small.

**Proof.** Let $N < M$. Then there exists ideal $I$ of $R$ such that $N = IM$. Since $R$ is local so that $J(R) = m$ and $I \subseteq m$. Now by Lemma 2.3 $N \subseteq J(M)$ and by Lemma 2.6, $N$ is small. \(\square\)

**Corollary 2.8.** Let $R$ be a local ring and $M$ a multiplication $R$-module. Then $M$ satisfy the following statements.

(i) $J(M) = J(R)M$.

(ii) $M$ is local.

(iii) every submodule of $M$ is small.

**Proof.** (i) By 2.3.

(ii) Let $J(R) = m$ and $N_1 = I_1M$, $N_2 = I_2M$ maximal submodules of $M$. Then $N_1 \subseteq J(R)M$ and $N_2 \subseteq J(R)M$. Since $N_1$ and $N_2$ are maximal so that $N_1 = N_2 = J(R)M$.

(iii) By 2.7. \(\square\)

**Theorem 2.9.** Let $M$ be a multiplication module, $B < M$ and $J(M/B) = 0$. Then $J(M) \subseteq B$.

**Proof.** Let $x \notin B$. Since $J(M/B) = 0$ by Lemma 2.2 there exists maximal submodule $K$ of $M$ such that $x \notin K$ so that $x \notin J(M)$.

**Corollary 2.10.** Let $R$ be a local ring and $0 \neq M$ be a finitely generated multiplication $R$-module. Then $M$ is a cyclic module.

**Proof.** Let $N$ be a maximal submodule of $M$ and $a \in M - N$. But $< a > + N = M$. Now since by Theorem 2.7 $N$ is small so that $< a > = M$. \(\square\)

**Lemma 2.11.** Let $M$ be a finitely generated $R$-module, say $M = \langle \{a_i\}_{i=1}^n \rangle$. If $x \in J(M)$, and $r \in R$, then $M = \langle a_k - rx, \{a_i\}_{i \neq k} \rangle$.

**Proof.** See [10], Lemma 2.1. \(\square\)

**Theorem 2.12.** Let $M$ be a finitely generated (not necessarily multiplication) module, $N$ is small in $M$ iff $\frac{M}{N} = \langle \{m_i + N\}_{i=1}^n \rangle$, implies that $M = \langle \{m_i\}_{i=1}^n \rangle$.

**Proof.** ($\implies$) By Lemma 2.6 $N \subseteq J(M)$. Now let $M = \langle g_1, g_2, \ldots, g_m \rangle$ and $\frac{M}{N} = \langle \{m_i + N\}_{i=1}^n \rangle$ so for each $g_k$ we have $g_k + N = \sum_{i=1}^n r_i^{(k)} m_i + N$ for each $k = 1, 2, \ldots, m$. Hence $g_k = \sum r_i^{(k)} m_i + n_k$ for some $n_k \in N \subseteq J(M)$. Now by Lemma 2.11, $M = \langle g_1 - n_1, \ldots, g_m - n_m \rangle$ and for each $k$, $g_k - n_k \in \langle \{m_i\}_{i=1}^n \rangle$. 

(⇐) Let \( M = \langle g_1, g_2, \ldots, g_n \rangle \) and \( N + K = M \) we shall show that \( K = M \). We have \( g_i = n_i + k_i \), \( 1 \leq i \leq n \) for some \( n_i \in N, k_i \in K \). So that \( \langle g_1 - n_1, g_2 - n_2, \ldots, g_n - n_n \rangle \leq K \). But

\[
\frac{M}{N} = \langle g_1 + N, \ldots, g_n + N \rangle = \langle g_1 - n_1 + N, \ldots, g_n - n_n + N \rangle.
\]

Now by assumption \( M = \langle g_1 - n_1, \ldots, g_n - n_n \rangle \). Hence \( K = M \). \( \square \)

**Corollary 2.13.** Let \( M \) be a finitely generated \( R \)-module and \( N \) small in \( M \). Then \( \text{rank} \, M = \text{rank} \, M_N \).

**Proof.** By Lemma 2.6 \( N \subseteq J(M) \). Now let \( \text{rank} \, M = m \) and \( \text{rank} \, \frac{M}{N} = n \). Clearly \( m \geq n \). If \( \{m_1 + N, \ldots, m_n + N\} \) is a minimal generating set of \( \frac{M}{N} \). Then by Theorem 2.12 we have that \( M = \langle m_1, \ldots, m_n \rangle \) which implies that \( \text{rank} \, M = m \leq n \). Thus \( \text{rank} \, M = \text{rank} \, \frac{M}{N} \). \( \square \)

**Theorem 2.14.** Let \( M \) be a finitely generated \( R \)-module and \( N \leq M \). Then the following statements are equivalent:

(i) \( N \) is small in \( M \)

(ii) \( N \subseteq J(M) \)

(iii) If \( \frac{M}{N} = \langle \{m_i + N\}_{i=1}^n \rangle \), then \( M = \langle m_i \rangle_{i=1}^n \).

**Proof.** By 2.6 and 2.12. \( \square \)

**Theorem 2.15.** Let \( M \) be a finitely generated \( R \)-module, \( N < M \) and

\[
\frac{M}{N} = \langle \{m_i + N\}_{i=1}^n \rangle \implies M = \langle m_i \rangle_{i=1}^n.
\]

Then \( M \) is local and so \( N \) is small.

**Proof.** Let \( N_1 \) and \( N_2 \) be maximal submodules of \( M \) and \( a \in N_1 - N_2 \). Then since \( \frac{M}{N_2} \) is simple module \( \frac{M}{N_2} = \langle a + N_2 \rangle \) therefore by assumption \( M = \langle a \rangle \) that is a contradiction because \( \langle a \rangle \subseteq N_1 \). Hence \( J(M) = N_1 \). But \( N \subseteq N_1 = J(M) \) consequently \( N \) is small. \( \square \)

**Theorem 2.16.** Let \( M \) be a finitely generated or multiplication \( R \)-module such that every prime submodule of \( M \) is small. Then \( M \) is local and every submodule of \( M \) is small and also \( M \) is cyclic module.

**Proof.** Since every maximal submodule is prime and so that small. Therefore if \( N_1 \) and \( N_2 \) are maximal submodules. Hence \( N_1 + N_2 = M \) and therefore \( N_1 = M \) or \( N_2 = M \) that is a contradiction consequently \( M \) is local say maximal submodule is \( N_1 \). Now since every proper submodule \( N \) is contain in \( N_1 = J(M) \) so that \( N \) is small. Now let \( a \in M - N_1 \). Then \( N_1 + \langle a \rangle = M \). Since \( N_1 \) is small so that \( \langle a \rangle = M \). \( \square \)
References


