ON PERTURBATION PROPERTIES OF FUZZY RELATION EQUATIONS WITH MAX-PRODUCT COMPOSITION

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Abstract: In this paper, the solving process of fuzzy relation equations with max-product composition is simplified. By the fuzzy solution invariant matrices, the perturbation properties of fuzzy relation equations with max-product composition are considered.

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The perturbation techniques is an important mathematical tool and the perturbation theory of fuzzy relation equations to be applied to fuzzy control, fuzzy inference and fuzzy logic as well. Tang [6] has discussed the perturbation issues of fuzzy relation equations with max-min composition. In this paper, the solution invariant matrix of fuzzy matrix is defined and the solving process of
fuzzy relation equations with max-product composition is simplified. By the fuzzy solution invariant matrices, the perturbation properties of fuzzy relation equations are considered.

1. Basic Notions

Let

\[ A \circ X = B, \quad \bigvee_{j=1}^{n} (a_{ij}x_j) = b_i \quad (i = 1, 2, \ldots, m) \]

be a fuzzy relation equation, where

\[ A = (a_{ij})_{n \times m}, \quad X = (x_j)_{n \times 1}, \quad B = (b_i)_{m \times 1} \]

are fuzzy matrices with elements belong to \([0, 1]\) and the sign “\(\circ\)” stands for the max-product composition.

Similar to the paper [6], the perturbation elements in matrix \(A\) can be defined as follows.

**Definition 1.1.** In equation (1), assume that \(a_{ij}\) is an element of \(A\). If for \(\varepsilon > 0\), where \(\varepsilon\) is small enough, such that \(a_{ij} - \varepsilon > 0, a_{ij} + \varepsilon \leq 1\), when \(a_{ij}\) perturbs in \([a_{ij} - \varepsilon, a_{ij} + \varepsilon]\), the set of all solution of the equation (1) varies, then \(a_{ij}\) is called an element without perturbation in \(A\), denoted by EWP.

**Definition 1.2.** In equation (1), assume that \(a_{ij}\) is an element of \(A\). If \(a_{ij}\) perturbs within \([a_{ij}', 1]\)(\(a_{ij} \leq a_{ij}'\)), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called an upper-closed middle perturbation element in \(A\), denoted by UCMPE.

**Definition 1.3.** In equation (1), assume that \(a_{ij}\) is an element of \(A\). If \(a_{ij}\) perturbs within \([0, a_{ij}']\)(\(a_{ij} \geq a_{ij}'\)), the set of all solution of the equation (1) is invariable, then \(a_{ij}\) is called an upper-closed perturbation element in \(A\), denoted by UCPE.
equation (1) is invariable, then \( a_{ij} \) is called a lower-closed perturbation element in \( A \), denoted by LCPE. If \( a_{ij} \) is not only an LCMPE but also a UCMPE, then \( a_{ij} \) is called a MCPE.

### 2. Fuzzy Solution Invariant Matrix

The solving process of fuzzy relation equation

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
& \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\circ
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix}
\]

has been studied by some scholars [5], [3], [2], [4], [1]. We have given the following simplifying by the fuzzy solution invariant matrix.

**Definition 2.1.** Let \( \beta : [0, 1]^2 \to [0, 1] \) be a mapping, \( \forall a, b \in [0, 1] \), define

\[
a_{\beta}b = \begin{cases} 
b/a, & \text{if } a > b, \\
1, & \text{if } a \leq b. 
\end{cases}
\]

For the equation (1), we have the following result.

**Lemma 2.1.** If the equation (1) is solvable, then

\[
\bigwedge_{j=1}^{n} a_{ij} \geq b_i, \quad i = 1, 2, \ldots, m.
\]

**Lemma 2.2.** The equation (1) is solvable iff \( X_d = A^T \beta B \) is the maximum solution of the equation (1), where \( X_d = (c_1, c_2, \ldots, c_n)^T \), and \( c_j = \bigwedge_{i=1}^{m} a_{ij} b_i \).

**Definition 2.2.** If the equation (1) is solvable, then \( S(A, B) = X_0 | A \circ X_0 = B \) is called the set of all solution of the equation (1). For \( X_1, X_2 \in S(A, B) \), \( X_i = (x_1^i, x_2^i, \ldots, x_n^i) (i = 1, 2) \), let \( X_1 \leq X_2 \) iff \( x_k^1 \leq x_k^2 (k = 1, 2, \ldots, n) \). It is obvious that \( \leq \) is a partial ordering on \( S(A, B) \), \( (S(A, B), \leq) \) is a lattice with min and max as its meet and join, respectively.

In the following, we suppose \( S(A, B) \neq \emptyset \).

**Definition 2.3.** Given the equation (1), then

\[
A^{(1)} = (a_{ij}^{(1)}), a_{ij}^{(1)} = \begin{cases} 
a_{ij}, & a_{ij} c_j = b_i, \\
0, & \text{otherwise}. 
\end{cases}
\]
is called the reduced matrix of $A$, where $X_d = (c_1, c_2, \ldots, c_n)^T$ is the maximum solution of the equation (1).

**Theorem 2.3.** Given the equation (1) then

$$S(A, B) = S(A^{(1)}, B).$$

**Proof.** Suppose that $X_d = (c_1, c_2, \ldots, c_n)^T$ is the maximum solution of the equation (1). $\forall i \in \{1, 2, \ldots, m\}$, we denote $I_1^i = \{j|a_{ij}c_j < b_i\}$, $I_2^i = \{j|a_{ij}c_j = b_i\}$, $I_3^i = \{j|a_{ij}c_j > b_i\}$. For brevity, $I_1^i$, $I_2^i$, $I_3^i$ are denoted by $I_1$, $I_2$, $I_3$ respectively. If there exists a pair $(i, j)$ such that $a_{ij}c_j > b_i$, then $\bigvee_{j=1}^n (a_{ij}c_j) \geq b_i$, it implies $I_3 = \emptyset$.

Let $X = (x_1, x_2, \ldots, x_n)^T \in S(A, B)$, then for all $i$, we have

$$b_i = \bigvee_{j=1}^n (a_{ij}x_j) = (\bigvee_{j \in I_1} (a_{ij}x_j)) \bigvee (\bigvee_{j \in I_2} (a_{ij}x_j)).$$

Since $\bigvee_{j \in I_1} (a_{ij}x_j) < b_i$, thus $b_i = \bigvee_{j \in I_1} (a_{ij}x_j) = \bigvee_{j=1}^n (a_{ij}^{(1)}x_j)$. Then $X = (x_1, x_2, \ldots, x_n)^T \in S(A^{(1)}, B)$.

Conversely, suppose that $J_1 = \{j|a_{ij}^{(1)} = 0\}$, $J_2 = \{j|a_{ij}^{(1)} \neq 0\}$, then $J_1 = I_1 \cup I_3 = I_1$, $J_2 = I_2$, hence

$$c_j = \bigwedge_{i=1}^m (a_{ij}\beta b_i) = (\bigwedge_{j \in J_1} (a_{ij}\beta b_i)) \bigwedge (\bigwedge_{j \in J_2} (a_{ij}\beta b_i)).$$

If $j \in J_1$, and $a_{ij} \leq b_i$, then $a_{ij}\beta b_i = a_{ij}^{(1)}\beta b_i = 1$.

If $j \in J_1$, $a_{ij} > b_i$, and $a_{ij}c_j \neq b_i$, then $a_{ij}c_j < b_i$, $a_{ij}\beta b_i = b_i/a_{ij} > c_j$, thus

$$c_j = \bigwedge_{j \in J_2} (a_{ij}\beta b_i) = \bigwedge_{i=1}^m (a_{ij}^{(1)}\beta b_i).$$

It illustrates that the maximum solution of $A \circ X = B$ is equal to the ones of $A^{(1)} \circ X = B$.

Suppose $X = (x_1, x_2, \ldots, x_n)^T \in S(A^{(1)}, B)$, we have

$$b_i = \bigvee_{j=1}^n (a_{ij}^{(1)}x_j) \leq \bigvee_{j=1}^n (a_{ij}x_j) \leq \bigvee_{j=1}^n (a_{ij}c_j) = b_i,$$
then \( X \in S(A, B) \), it implies that \( S(A, B) = S(A^{(1)}, B) \).

**Definition 2.4.** Let \( A \circ X = B \) and \( E \circ X = B \) be two fuzzy relation equations. \( A \) and \( E \) are called the fuzzy solution invariant matrices about \( B \) if \( S(A, B) = S(E, B) \).

**Theorem 2.4.** Let \( A \circ X = B \) and \( E \circ X = B \) be two fuzzy relation equations. \( A \) and \( E \) are fuzzy solution invariant matrices about \( B \) iff \( S(A^{(1)}, B) = S(E^{(1)}, B) \).

**Proof.** It is clear by Theorem 2.3 and Definition 2.4.

3. The Perturbation Issues of Fuzzy Relation Equations

**Definition 3.1.** Let \( A = (a_{ij})_{m \times n} \) and \( C = (c_{ij})_{m \times n} \) be two fuzzy matrices. We shall write \( A \leq C \) if \( a_{ij} \leq c_{ij} \) for all pairs \((i, j)\), where \( 1 \leq i \leq m, 1 \leq j \leq n \). And write \( A < C \) if \( A \leq C \) and \( A \neq B \).

**Definition 3.2.** Given the equation (1). \( A \) is called a fuzzy matrix without perturbation about \( B \) if every element \( a_{ij} \) of \( A \) is the EWP of \( A \). Otherwise, \( A \) is called a fuzzy perturbation matrix.

**Theorem 3.1.** Given the equation (1). Then \( a_{ij} \) is a LPE of \( A \) within \([0, b_i] \) if \( a_{ij} < b_i \). Moreover, \( a_{ij} \) is a LCPE of \( A \) within \([0, b_i] \) when \( c_j < 1 \). \( a_{ij} \) is a LPE of \( A \) within \([0, b_i] \) and is not a LCPE of \( A \) within \([0, b_i] \) when \( c_j = 1 \).

**Proof.** If \( a_{ij} < b_i \), then \( a_{ij} \beta b_i = 1 \) and \( a_{ij}^{(1)} = 0 \). We replace \( a_{ij} \) by \( e_{ij} \left( e_{ij} \in [0, b_i] \right) \) and let the other elements are invariable, so we get a new matrix \( E = (e_{ij}) \). Hence \( e_{ij} \beta b_i = 1 \) and \( e_{ij}^{(1)} = 0 \), thus \( E^T \beta B = A^T \beta B \) and \( E^{(1)} = A^{(1)} \). From Theorem 2.4, \( S(E, B) = S(A, B) \), \( a_{ij} \) is a LPE of \( A \) within \([0, b_i] \).

Moreover, when \( c_j < 1 \), for above the matrix \( E \), if \( e_{ij} = b_i \), then \( e_{ij} \beta b_i = 1 \) and \( e_{ij}^{(1)} = 0 \). Thus \( E^T \beta B = A^T \beta B \) and \( E^{(1)} = A^{(1)} \). \( a_{ij} \) is a LCPE of \( A \) within \([0, b_i] \).

When \( c_j = 1 \), \( e_{ij}^{(1)} = b_i < 1 \), it implies that \( E^{(1)} \neq A^{(1)} \), so \( S(E, B) \neq S(A, B) \), \( a_{ij} \) is not a LCPE of \( A \) within \([0, b_i] \).

**Theorem 3.2.** Given the equation (1). If \( a_{ij} b_i \) and \( a_{ij}^{(1)} = 0 \), then \( a_{ij} \) is a LCPE of within \([0, b_i] \).

**Proof.** Assume that \( a_{ij} b_i \) and \( a_{ij}^{(1)} = 0 \), then \( a_{ij} c_j < b_i \), so \( c_j < 1 \). We replace \( a_{ij} \) by \( e_{ij} \left( e_{ij} \in [0, b_i] \right) \) and let the other elements be invariable, so we get a new matrix \( E = (e_{ij}) \). Then \( e_{ij} \beta b_i = a_{ij} \beta b_i = 1 \) and \( e_{ij}^{(1)} = a_{ij}^{(1)} = 0 \), thus
$ET\beta B = A^T\beta B$ and $E^{(1)} = A^{(1)}$, it implies that $S(E, B) = S(A, B)$, $a_{ij}$ is a LCPE of $A$ within $[0, b_i]$. \hfill \Box

**Theorem 3.3.** Given the equation (1) and assume that $a_{ij} > b_i$ and $a_{ij}^{(1)} = 0$, then $a_{ij}$ is a LCMPE of $A$ within $[b_i, b_i/c_j]$.

**Proof.** Assume that $a_{ij} > b_i$ and $a_{ij}^{(1)} = 0$, then $b_i/a_{ij} = a_{ij}\beta b_i > \bigwedge_{t=1}^{m} (a_{tij}\beta b_i) = c_j$. We replace $a_{ij}$ by $e_{ij}([b_i, b_i/c_j])$, and let the other elements be invariable, so we get a new matrix $E = (e_{ij})$. Then $b_i/e_{ij} \in (c_j, 1]$, $b_i/e_{ij} = e_{ij}\beta b_i > c_j$.

Hence

$$c_j = \bigwedge_{t=1}^{m} a_{ij}\beta b_t = (\bigwedge_{t\neq i} a_{tij}\beta b_t) \bigwedge_{t\neq i} (a_{tij}\beta b_t) = \bigwedge_{t\neq i} a_{tij}\beta b_t$$

$$= (\bigwedge_{t\neq i} a_{tij}\beta b_t) \bigwedge_{t\neq i} (e_{ij}\beta b_t) = (\bigwedge_{t\neq i} e_{tij}\beta b_t) \bigwedge_{t\neq i} (e_{tij}\beta b_t) = \bigwedge_{t=1}^{m} e_{tij}\beta b_t.$$ 

So $A^T\beta B = ET\beta B$. Moreover, $e^{(1)}_{ij} = 0 = a^{(1)}_{ij}$, $E^{(1)} = A^{(1)}$, by Theorem 2.4, $S(E, B) = S(A, B)$, $a_{ij}$ is a LCMPE of $A$ within $[b_i, b_i/c_j]$. \hfill \Box

**Theorem 3.4.** Given the equation (1), if $a_{ij}^{(1)} \neq 0$, then $a_{ij}$ is an EWP of $A$.

**Proof.** Suppose $a_{ij}^{(1)} \neq 0$, then $a_{ij}c_j = b_i$,

$$c_j = b_i/a_{ij} = a_{ij}\beta b_i = (\bigwedge_{t\neq i} a_{tij}\beta b_t) \bigwedge_{t\neq i} (a_{tij}\beta b_t),$$

thus $a_{ij}\beta b_i = (\bigwedge_{t\neq i} a_{tij}\beta b_t)$. If we replace $a_{ij}$ by $e_{ij}(e_{ij} \in [0, b_i] \cup (b_i/c_j, 1])$, and let the other elements be invariable, so we get a new matrix $E = (e_{ij})$. Then $e^{(1)}_{ij} = 0 \neq e^{(1)}_{ij}$, $E^{(1)} \neq A^{(1)}$ it implies that $S(E, B) \neq S(A, B)$.

Thus $a_{ij}$ is an EWP of $A$. \hfill \Box

**Corollary 3.5.** Let $a_{ij}^{(1)} \neq 0$ and $a_{ij}$ be an EWP of $A$, if $a_{ij}$ perturbs within $[0, b_i]$ and let the other elements be invariable, so we get a new matrix $E$. Then $S(E, B) \subset S(A, B)$.

**Corollary 3.6.** Let $a_{ij}^{(1)} \neq 0$ and $a_{ij}$ be an EWP of $A$, if $a_{ij}$ perturbs within $(b_i/c_j, 1]$ and let the other elements are invariable, so we get a new matrix $E$. Then $S(E, B) \subset S(A, B)$.

**Theorem 3.7.** Given the equation (1), $A$ is a fuzzy matrix without perturbation about $B$ if and only if $A = A^{(1)}$. 


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References


