International Journal of Pure and Applied Mathematics

Volume 17 No. 2 2004, 217-228

INTUITIONISTIC FUZZY THEORY OF IDEALS IN GAMMA-NEAR-RINGS

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Abstract: We introduce the notion of intuitionistic fuzzy ideals in Γ-near-rings, and then some basic properties are investigated. Characterizations of intuitionistic fuzzy ideals are given. Using a collection of ideals, an intuitionistic fuzzy ideal is established. The notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a Γ-near-ring are discussed.

AMS Subject Classification: 16Y30, 16Y99, 03E72, 03F55
Key Words: Γ-near-ring, intuitionistic fuzzy ideal

1. Introduction

Γ-near-rings were defined by Satyanarayana [5], and the ideal theory in Γ-near-

Received: October 15, 2004

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rings was studied by Satyanarayana [5] and Booth [3]. Jun et al [4] discussed the fuzzification of ideals in Γ-near-rings. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. In this paper, using the Atanassov's idea, we establish the intuitionistic fuzzification of the concept of ideals in Γ-near-rings, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a Γ-near-ring and investigate some related properties.

2. Preliminaries

We first recall some basic concepts for the sake of completeness. A non-empty set R with two binary operations "+"(addition) and "$\cdot$" (multiplication) is called a *near-ring* if it satisfies the following axioms:

- $(R, +)$ is a group,
- $(R, \cdot)$ is a semigroup,
- $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a right near-ring because it satisfies the right distributive law. We will use the word "near-ring" to mean "right near-ring". We denote xy instead of $x \cdot y$.

A Γ-*near-ring* is a triple $(M, +, \Gamma)$, where:

- $(M, +)$ is a group,
- Γ is a nonempty set of binary operators on M such that for each $\alpha \in \Gamma$, $(M, +, \alpha)$ is a near-ring,
- $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

A nonempty subset J of a Γ-near-ring M is called an *ideal* of M if it satisfies:
(I1) $(J, +)$ is a normal divisor of $(M, +)$, that is,

- $x - y \in J, \quad \forall x, y \in J,$
- $y + x - y \in J, \quad \forall x \in J, \quad \forall y \in M.$

(I2) $u\alpha(x + v) - u\alpha v \in J, \quad \forall x \in J, \quad \forall \alpha \in \Gamma, \quad \forall u, v \in M.$

We now review some fuzzy logic concepts. A fuzzy set in a set M is a function $\mu : M \to [0, 1]$. We shall use the notation $U(\mu; t)$ (resp. $L(\mu; t)$), called a *upper* (resp. *lower*) *level subset* of μ, for $\{x \in M \mid \mu(x) \geq t\}$ (resp. $\{x \in M \mid \mu(x) \leq t\}$), where $t \in [0, 1]$.

A fuzzy set μ in a Γ-near-ring M is called a *fuzzy ideal* of M (see [4]) if it satisfies:

(F1) μ is a fuzzy normal divisor with respect to the addition, that is,

- $\mu(x-y) \geq \min\{\mu(x), \quad \mu(y)\}, \quad \forall x, y \in M.$
- $\mu(y+x-y) \geq \mu(x), \quad \forall x, y \in M.$

(F2) $\mu\Big(u\alpha(x+v) - u\alpha v\Big) \geq \mu(x), \quad \forall x, u, v \in M, \quad \forall \alpha \in \Gamma.$

Let X be a non-empty fixed set. An *intuitionistic fuzzy set* (IFS for short) A is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\},$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ (see Atanassov [1, 2]).

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

3. Intuitionistic Fuzzy Ideals

In what follows, let M denotes a Γ-near-ring unless otherwise specified.

Definition 3.1. An IFS $A = (\mu_A, \gamma_A)$ in M is called an *intuitionistic fuzzy ideal* of M if

(IF1) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal divisor with respect to the addition,

(IF2) $\mu_A\Big(u\alpha(x+v) - u\alpha v\Big) \geq \mu_A(x)$ and $\gamma_A\Big(u\alpha(x+v) - u\alpha v\Big) \leq \gamma_A(x)$ for all $x, u, v \in M$ and $\alpha \in \Gamma$.

The condition (IF1) means that $A = (\mu_A, \gamma_A)$ satisfies:

- $\mu_A(x-y) \geq \min\Big\{\mu_A(x), \quad \mu_A(y)\Big\}$ and $\mu_A(y+x-y) \geq \mu_A(x)$,
- $\gamma_A(x-y) \leq \max\Big\{\gamma_A(x), \quad \gamma_A(y)\Big\}$ and $\gamma_A(y+x-y) \leq \gamma_A(x)$,

for all $x, y \in M$.

Note that if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M, then

$$\mu_A(0) \geq \mu_A(x) = \mu_A(-x) \quad \text{and} \quad \gamma_A(0) \leq \gamma_A(x) = \gamma_A(-x)$$

for all $x \in M$, where 0 is the zero element of M.

Theorem 3.2. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M, then the sets*

$$M_{\mu_A} := \{x \in M \mid \mu_A(x) = \mu_A(0)\} \text{ and } M_{\gamma_A} := \{x \in M \mid \gamma_A(x) = \gamma_A(0)\}$$

are ideals of M.

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M and let $x, y \in M_{\mu_A} \cap M_{\gamma_A}$. Then

$$\mu_A(x - y) \geq \min\{\mu_A(x), \quad \mu_A(y)\} = \mu_A(0),$$

$$\gamma_A(x - y) \leq \max\{\gamma_A(x), \quad \gamma_A(y)\} = \gamma_A(0),$$

and so $\mu_A(x - y) = \mu_A(0)$ and $\gamma_A(x - y) = \gamma_A(0)$. Hence $x - y \in M_{\mu_A} \cap M_{\gamma_A}$. For every $y \in M$ and $x \in M_{\mu_A} \cap M_{\gamma_A}$, we have

$$\mu_A(y + x - y) \geq \mu_A(x) = \mu_A(0) \quad \text{and} \quad \gamma_A(y + x - y) \leq \gamma_A(x) = \gamma_A(0).$$

It follows that $\mu_A(y + x - y) = \mu_A(0)$ and $\gamma_A(y + x - y) = \gamma_A(0)$, that is, $y + x - y \in M_{\mu_A} \cap M_{\gamma_A}$. Thus M_{μ_A} and M_{γ_A} are normal divisors of M with respect to the addition. Let $x \in M_{\mu_A} \cap M_{\gamma_A}$, $\alpha \in \Gamma$ and $u, v \in M$. Then

$$\mu_A\Big(u\alpha(x + v) - u\alpha v\Big) \geq \mu_A(x) = \mu_A(0),$$

$$\gamma_A\Big(u\alpha(x + v) - u\alpha v\Big) \leq \gamma_A(x) = \gamma_A(0),$$

and thus $\mu_A\Big(u\alpha(x + v) - u\alpha v\Big) = \mu_A(0)$ and $\gamma_A\Big(u\alpha(x + v) - u\alpha v\Big) = \gamma_A(0)$. Therefore $u\alpha(x + v) - u\alpha v \in M_{\mu_A} \cap M_{\gamma_A}$, and consequently M_{μ_A} and M_{γ_A} are ideals of M. □

Theorem 3.3. *Let J be an ideal of M and let $A = (\mu_A, \gamma_A)$ be an IFS in M defined by*

$$\mu_A(x) := \begin{cases} t_0 & \text{if} \quad x \in J, \\ t_1 & \text{otherwise,} \end{cases} \qquad \gamma_A(x) := \begin{cases} s_0 & \text{if} \quad x \in J, \\ s_1 & \text{otherwise,} \end{cases}$$

for all $x \in M$ *and* $s_i, t_i \in [0,1]$ *such that* $t_0 > t_1$, $s_0 < s_1$ *and* $s_i + t_i \leq 1$ *for* $i = 0, 1$. *Then* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal of* M, *and* $U(\mu_A; t_0) = J = L(\gamma_A; s_0)$.

Proof. Let $x, y \in M$. If $x, y \in J$, then $x - y \in J$ and so

$$\mu_A(x-y) = t_0 = \min\{\mu_A(x), \quad \mu_A(y)\},$$
$$\gamma_A(x-y) = s_0 = \max\{\gamma_A(x), \quad \gamma_A(y)\}.$$

If any one of x and y does not belong to J, then

$$\mu_A(x-y) \geq t_1 = \min\{\mu_A(x), \quad \mu_A(y)\},$$
$$\gamma_A(x-y) \leq s_1 = \max\{\gamma_A(x), \quad \gamma_A(y)\}.$$

If $x \notin J$, then

$$\mu_A(y + x - y) \geq t_1 = \mu_A(x), \qquad \gamma_A(y + x - y) \leq s_1 = \gamma_A(x).$$

If $x \in J$, then $y + x - y \in J$ and thus

$$\mu_A(y + x - y) = t_0 = \mu_A(x), \qquad \gamma_A(y + x - y) = s_0 = \gamma_A(x).$$

Let $x, u, v \in M$ and $\alpha \in \Gamma$. If $x \notin J$, then

$$\mu_A(u\alpha(x+v) - u\alpha v) \geq t_1 = \mu_A(x), \qquad \gamma_A(u\alpha(x+v) - u\alpha v) \leq s_1 = \gamma_A(x).$$

Assume that $x \in J$. Then $u\alpha(x+v) - u\alpha v \in J$ by (I2). Hence

$$\mu_A(u\alpha(x+v) - u\alpha v) = t_0 = \mu_A(x), \qquad \gamma_A(u\alpha(x+v) - u\alpha v) = s_0 = \gamma_A(x).$$

Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M. Obviously, $U(\mu_A; t_0) = J = L(\gamma_A; s_0)$. □

Corollary 3.4. *Any ideal of* M *can be realized as both an upper level ideal and a lower level ideal of some intuitionistic fuzzy ideal of* M.

Proof. The proof is straightforward. □

Theorem 3.5. *If an IFS* $A = (\mu_A, \gamma_A)$ *in* M *is an intuitionistic fuzzy ideal of* M, *then the nonempty upper and lower level sets* $U(\mu_A; t)$ *and* $L(\gamma_A, t)$ *of* A *are ideals of* M *for every* $t \in [0,1]$.

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M and let $t \in [0,1]$ be such that $U(\mu_A; t) \neq \emptyset$ and $L(\gamma_A, t) \neq \emptyset$. Let $x, y \in U(\mu_A; t) \cap L(\gamma_A; t)$. Then

$$\mu_A(x-y) \geq \min\Big\{\mu_A(x), \quad \mu_A(y)\Big\} \geq t,$$

$$\gamma_A(x-y) \leq \max\Big\{\gamma_A(x), \quad \gamma_A(y)\Big\} \leq t,$$

and thus $x-y \in U(\mu_A;t) \cap L(\gamma_A;t)$. Now let $x \in U(\mu_A;t) \cap L(\gamma_A;t)$ and $y \in M$. Then $\mu_A(x) \geq t$, $\gamma_A(x) \geq t$, and so

$$\mu_A(y+x-y) \geq \mu_A(x) \geq t, \qquad \gamma_A(y+x-y) \leq \gamma_A(x) \leq t.$$

Hence $y+x-y \in U(\mu_A;t) \cap L(\gamma_A;t)$.

Finally, let $x \in U(\mu_A;t) \cap L(\gamma_A;t)$, $\alpha \in \Gamma$, and $u, v \in M$. Using (IF2), we get

$$\mu_A\Big(u\alpha(x+v) - u\alpha v\Big) \geq \mu_A(x) \geq t,$$

$$\gamma_A\Big(u\alpha(x+v) - u\alpha v\Big) \leq \gamma_A(x) \leq t,$$

which imply that $u\alpha(x+v) - u\alpha v \in U(\mu_A;t) \cap L(\gamma_A;t)$. Therefore $U(\mu_A;t)$ and $L(\gamma_A;t)$ are ideals of M. □

We consider the converse of Theorem 3.5.

Theorem 3.6. *Let $A = (\mu_A, \gamma_A)$ be an IFS in M such that the nonempty upper and lower level sets $U(\mu_A;t)$ and $L(\gamma_A,t)$ of A are ideals of M for every $t \in [0,1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M.*

Proof. Let $t \in [0,1]$ and suppose that $U(\mu_A;t)(\neq \emptyset)$ and $L(\gamma_A,t)(\neq \emptyset)$ are ideals of M. Assume that there are $x_0, y_0 \in M$ such that

$$\mu_A(x_0-y_0) < \min\Big\{\mu_A(x_0), \ \mu_A(y_0)\Big\},$$

$$\gamma_A(x_0-y_0) > \max\Big\{\gamma_A(x_0), \ \gamma_A(y_0)\Big\}.$$

Taking

$$p_0 := \frac{1}{2}\Big(\mu_A(x_0-y_0) + \min\{\mu_A(x_0), \ \mu_A(y_0)\}\Big),$$

$$q_0 := \frac{1}{2}\Big(\gamma_A(x_0-y_0) + \max\{\gamma_A(x_0), \ \gamma_A(y_0)\}\Big),$$

then

$$\mu_A(x_0-y_0) < p_0 < \min\{\mu_A(x_0), \quad \mu_A(y_0)\},$$

$$\gamma_A(x_0-y_0) > q_0 > \max\{\gamma_A(x_0), \quad \gamma_A(y_0)\}.$$

It follows that $x_0, y_0 \in U(\mu_A;p_0) \cap L(\gamma_A;q_0)$, but

$$x_0 - y_0 \notin U(\mu_A;p_0) \cap L(\gamma_A;q_0).$$

This is impossible, and so

$$\mu_A(x-y) \geq \min\{\mu_A(x),\ \mu_A(y)\}, \quad \gamma_A(x-y) \leq \max\{\gamma_A(x),\ \gamma_A(y)\}, \quad \forall x, y \in M.$$

If there are $x_1, y_1 \in M$ such that

$$\mu_A(y_1 + x_1 - y_1) < \mu_A(x_1), \qquad \gamma_A(y_1 + x_1 - y_1) > \gamma_A(x_1),$$

then by taking

$$p_1 := \frac{1}{2}\Big(\mu_A(y_1 + x_1 - y_1) + \mu_A(x_1)\Big),$$
$$q_1 := \frac{1}{2}\Big(\gamma_A(y_1 + x_1 - y_1) + \gamma_A(x_1)\Big)$$

we have

$$\mu_A(y_1 + x_1 - y_1) < p_1 < \mu_A(x_1), \qquad \gamma_A(y_1 + x_1 - y_1) > q_1 > \gamma_A(x_1).$$

Hence $x_1 \in U(\mu_A; p_1) \cap L(\gamma_A; q_1)$ and $y_1 + x_1 - y_1 \notin U(\mu_A; p_1) \cap L(\gamma_A; q_1)$. This is a contradiction, so

$$\mu_A(y + x - y) \geq \mu_A(x), \qquad \gamma_A(y + x - y) \leq \gamma_A(x), \qquad \forall x, y \in M.$$

Finally, we show that $A = (\mu_A, \gamma_A)$ satisfies (IF2). If not, then there exist $x_2, u_2, v_2 \in M$ and $\beta \in \Gamma$ such that

$$\mu_A\Big(u_2\beta(x_2 + v_2) - u_2\beta v_2\Big) < \mu_A(x_2), \quad \gamma_A\Big(u_2\beta(x_2 + v_2) - u_2\beta v_2\Big) > \gamma_A(x_2).$$

Putting

$$p_2 := \frac{1}{2}\Big(\mu_A(u_2\beta(x_2 + v_2) - u_2\beta v_2) + \mu_A(x_2)\Big),$$
$$q_2 := \frac{1}{2}\Big(\gamma_A(u_2\beta(x_2 + v_2) - u_2\beta v_2) + \gamma_A(x_2)\Big),$$

we have

$$\mu_A\Big(u_2\beta(x_2 + v_2) - u_2\beta v_2\Big) < p_2 < \mu_A(x_2),$$
$$\gamma_A\Big(u_2\beta(x_2 + v_2) - u_2\beta v_2\Big) > q_2 > \gamma_A(x_2),$$

and thus $x_2 \in U(\mu_A; p_2) \cap L(\gamma_A; q_2)$ and

$$u_2\beta(x_2 + v_2) - u_2\beta v_2 \notin U(\mu_A; p_2) \cap L(\gamma_A; q_2).$$

This is impossible, and the proof is complete. □

Theorem 3.7. *If an IFS $A = (\mu_A, \gamma_A)$ in M is an intuitionistic fuzzy ideal of M, then so is $\mathbb{A} := \{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in M\}$.*

Proof. It is sufficient to show that $\bar{\mu}_A$ satisfies the second conditions of (IF1) and (IF2). For any $x, y \in M$, we have

$$\begin{aligned} \bar{\mu}_A(x - y) &= 1 - \mu_A(x - y) \leq 1 - \min\{\mu_A(x), \ \ \mu_A(y)\} \\ &= \max\{1 - \mu_A(x), \ \ 1 - \mu_A(y)\} \\ &= \max\{\bar{\mu}_A(x), \ \ \bar{\mu}_A(y)\}, \end{aligned}$$

$$\bar{\mu}_A(y + x - y) = 1 - \mu_A(y + x - y) \leq 1 - \mu_A(x) = \bar{\mu}_A(x).$$

Let $x, u, v \in M$ and $\alpha \in \Gamma$. Then

$$\bar{\mu}_A\Big(u\alpha(x + v) - u\alpha v\Big) = 1 - \mu_A\Big(u\alpha(x + v) - u\alpha v\Big) \leq 1 - \mu_A(x) = \bar{\mu}_A(x).$$

Hence $\mathbb{A}$ is an intuitionistic fuzzy ideal of M. □

Theorem 3.8. *Let $\{J_t \mid t \in \Lambda \subseteq [0, 1]\}$ be a collection of ideals of M such that*

(i) $M = \cup_{t\in\Lambda} J_t$,

(ii) *For every $s, t \in \Lambda$, $s > t$ if and only if $J_s \subsetneq J_t$.*

Define an IFS $A = (\mu_A, \gamma_A)$ in M by

$$\mu_A(x) := \sup\{t \in \Lambda \mid x \in J_t\}, \qquad \gamma_A(x) := \inf\{t \in \Lambda \mid x \in J_t\}, \qquad \forall x \in M.$$

Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M.

Proof. According to Theorem 3.6 it is sufficient to show that the nonempty upper and lower level sets $U(\mu_A; t)$ and $L(\gamma_A; l)$ of $A = (\mu_A, \gamma_A)$ are ideals of M for every $t, l \in [0, 1]$. In order to show that $U(\mu_A; t)$ is an ideal of M, we consider the following two cases:

(1°) $t = \sup\{s \in \Lambda \mid s < t\}$,

(2°) $t \neq \sup\{s \in \Lambda \mid s < t\}$.

For the case (1°) we get

$$x \in U(\mu_A;t) \quad \Leftrightarrow \quad x \in J_s \quad \forall s < t \quad \Leftrightarrow \quad x \in \cap_{s<t} J_s,$$

and so $U(\mu_A;t) = \cap_{s<t} J_s$, which is an ideal of M. Case (2°) implies that $U(\mu_A;t) = \cup_{s \geq t} J_s$, which is an ideal of M. In fact, if $x \in \cup_{s \geq t} J_s$ then $x \in J_s$ for some $s \geq t$. It follows that $\mu_A(x) \geq s \geq t$ so that $x \in U(\mu_A;t)$. Conversely, assume that $x \notin \cup_{s \geq t} J_s$. Then $x \notin J_s$ for all $s \geq t$. Since $t \neq \sup\{s \in \Lambda \mid s < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin J_s$ for all $s > t - \varepsilon$, which means that if $x \in J_s$ then $s \leq t - \varepsilon$. Thus $\mu_A(x) \leq t - \varepsilon < t$, and so $x \notin U(\mu_A;t)$. Now we prove that $L(\gamma_A;l)$ is an ideal of M. We consider the following two cases:

(3°) $l = \inf\{m \in \Lambda \mid l < m\}$,

(4°) $l \neq \inf\{m \in \Lambda \mid l < m\}$.

Case (3°) implies that

$$x \in L(\gamma_A;l) \quad \Leftrightarrow \quad x \in J_m \quad \forall m > l \quad \Leftrightarrow \quad x \in \cap_{m>l} J_m.$$

This shows that $L(\gamma_A;l)$ is an ideal of M.

Case (4°) implies that there exists $\varepsilon > 0$ such that $(l, l+\varepsilon) \cap \Lambda = \emptyset$. If $x \in \cup_{l \geq m} J_m$, then $x \in J_m$ for some $m \leq l$. It follows that $\gamma_A(x) \leq m \leq l$ so that $x \in L(\gamma_A;l)$. Hence $\cup_{l \geq m} J_m \subseteq L(\gamma_A;l)$. Now if $x \notin \cup_{l \geq m} J_m$, then $x \notin J_m$ for all $m \leq l$ which implies that $x \notin J_m$ for all $m < l + \varepsilon$, that is, if $x \in J_m$ then $m \geq l + \varepsilon$. Thus $\gamma_A(x) \geq l + \varepsilon > l$, that is, $x \notin L(\gamma_A;l)$. Therefore $L(\gamma_A;l) \subseteq \cup_{l \geq m} J_m$. This completes the proof. □

Theorem 3.9. *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of M if and only if μ_A and $\bar{\gamma}_A$ are fuzzy ideals of M.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of M. Then obviously μ_A is a fuzzy ideal of M. Let $x, y \in M$. Then

$$\begin{aligned} \bar{\gamma}_A(x-y) &= 1 - \gamma_A(x-y) \geq 1 - \max\{\gamma_A(x), \ \gamma_A(y)\} \\ &= \min\{1-\gamma_A(x), \ 1-\gamma_A(y)\} \\ &= \min\{\bar{\gamma}_A(x), \ \bar{\gamma}_A(y)\}, \end{aligned}$$

$$\bar{\gamma}_A(y+x-y) = 1 - \gamma_A(y+x-y) \geq 1 - \gamma_A(x) = \bar{\gamma}_A(x).$$

Hence $\bar{\gamma}_A$ is a fuzzy normal divisor with respect to the addition. Let $x, u, v \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \bar{\gamma}_A\Big(u\alpha(x+v) - u\alpha v\Big) &= 1 - \gamma_A\Big(u\alpha(x+v) - u\alpha v\Big) \\ &\geq 1 - \gamma_A(x) = \bar{\gamma}_A(x). \end{aligned}$$

Therefore $\bar{\gamma}_A$ is a fuzzy ideal of M. Conversely, suppose that μ_A and $\bar{\gamma}_A$ are fuzzy ideals of M. Let $x, y \in M$. Then

$$\begin{aligned} 1-\gamma_A(x-y) &= \bar{\gamma}_A(x-y) \\ &\geq \min\{\bar{\gamma}_A(x), \quad \bar{\gamma}_A(y)\} \\ &= \min\{1-\gamma_A(x), \quad 1-\gamma_A(y)\} \\ &= 1-\max\{\gamma_A(x), \quad \gamma_A(y)\}, \end{aligned}$$

$$1-\gamma_A(y+x-y) = \bar{\gamma}_A(y+x-y) \geq \bar{\gamma}_A(x) = 1-\gamma_A(x),$$

which imply that $\gamma_A(x-y) \leq \max\{\gamma_A(x), \quad \gamma_A(y)\}$ and $\gamma_A(y+x-y) \leq \gamma_A(x)$. Finally, for any $x, u, v \in M$ and $\alpha \in \Gamma$ we get

$$\begin{aligned} 1-\gamma_A\Big(u\alpha(x+v)-u\alpha v\Big) &= \bar{\gamma}_A\Big(u\alpha(x+v)-u\alpha v\Big) \\ &\geq \bar{\gamma}(x) = 1-\gamma_A(x), \end{aligned}$$

and hence $\gamma_A\Big(u\alpha(x+v)-u\alpha v\Big) \leq \gamma_A(x)$. This completes the proof. □

Let $\mathbb{F}(M)$ denote the family of all intuitionistic fuzzy ideals of M and let $t \in [0,1]$. Define binary relations $\sim_\mu$ and $\sim_\gamma$ on $\mathbb{F}(M)$ as follows:

$$A \sim_\mu B \quad \Leftrightarrow \quad U(\mu_A; t) = U(\mu_B; t),$$

$$A \sim_\gamma B \quad \Leftrightarrow \quad L(\gamma_A; t) = L(\gamma_B; t),$$

respectively, for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ in $\mathbb{F}(M)$. Then clearly $\sim_\mu$ and $\sim_\gamma$ are equivalence relations on $\mathbb{F}(M)$. For any $A = (\mu_A, \gamma_A) \in \mathbb{F}(M)$, let $[A]_\mu$ (resp. $[A]_\gamma$) denote the equivalence class of A modulo $\sim_\mu$ (resp. $\sim_\gamma$), and denote by $\mathbb{F}(M)/\sim_\mu$ (resp. $\mathbb{F}(M)/\sim_\gamma$) the collection of all equivalence classes of A modulo $\sim_\mu$ (resp. $\sim_\gamma$), that is,

$$\mathbb{F}(M)/\sim_\mu := \{[A]_\mu \mid A = (\mu_A, \gamma_A) \in \mathbb{F}(M)\}$$

$$(\text{resp. } \mathbb{F}(M)/\sim_\gamma := \{[A]_\gamma \mid A = (\mu_A, \gamma_A) \in \mathbb{F}(M)\}).$$

Now let $\mathbb{I}(M)$ denote the family of all ideals of M and let $t \in [0,1]$. Define maps f_t and g_t from $\mathbb{F}(M)$ to $\mathbb{I}(M) \cup \{\emptyset\}$ by

$$f_t(A) = U(\mu_A; t) \qquad \text{and} \qquad g_t(A) = L(\gamma_A; t),$$

respectively, for all $A = (\mu_A, \gamma_A) \in \mathbb{F}(M)$. Then f_t and g_t are clearly well-defined.

Theorem 3.10. *For any $t \in (0,1)$, the maps f_t and g_t are surjective from $\mathbb{F}(M)$ to $\mathbb{I}(M) \cup \{\emptyset\}$. Moreover the quotient sets $\mathbb{F}(M)/\sim_\mu$ and $\mathbb{F}(M)/\sim_\gamma$ are equipotent to $\mathbb{I}(M) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in $\mathbb{F}(M)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in M defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in M$. Obviously

$$f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \emptyset = L(\mathbf{1}; t) = g_t(\mathbf{0}_\sim).$$

Let $J(\neq \emptyset) \in \mathbb{I}(M)$. For $J_\sim = (\chi_J, \bar{\chi}_J) \in \mathbb{F}(M)$, we have

$$f_t(J_\sim) = U(\chi_J; t) = J = L(\bar{\chi}_J; t) = g_t(J_\sim).$$

Hence f_t and g_t are surjective. Let f_t^* (resp. g_t^*) be a map from $\mathbb{F}(M)/\sim_\mu$ (resp. $\mathbb{F}(M)/\sim_\gamma$) to $\mathbb{I}(M) \cup \{\emptyset\}$ defined by $f_t^*([A]_\mu) = f_t(A)$ (resp. $g_t^*([A]_\gamma) = g_t(A)$) for all $A = (\mu_A, \gamma_A) \in \mathbb{F}(M)$. Assume that $U(\mu_A; t) = U(\mu_B; t)$ and $L(\gamma_A; t) = L(\gamma_B; t)$ for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ in $\mathbb{F}(M)$. Then $A \sim_\mu B$ and $A \sim_\gamma B$, and hence $[A]_\mu = [B]_\mu$ and $[A]_\gamma = [B]_\gamma$. Therefore the maps f_t^* and g_t^* are injective. Now let $J(\neq \emptyset) \in \mathbb{I}(M)$. For $J_\sim = (\chi_J, \bar{\chi}_J) \in \mathbb{F}(M)$, we have

$$f_t^*([J_\sim]_\mu) = f_t(J_\sim) = U(\chi_J; t) = J = L(\bar{\chi}_J; t) = g_t(J_\sim) = g_t^*([J_\sim]_\gamma).$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathbb{F}(M)$, we get

$$f_t^*([0_\sim]_\mu) = f_t(0_\sim) = U(\mathbf{0}; t) = \emptyset = L(\mathbf{1}; t) = g_t(\mathbf{0}_\sim) = g_t^*([0_\sim]_\gamma).$$

This shows that f_t^* and g_t^* are surjective, and we are done. □

For any $t \in [0,1]$ we define another relation $\mathfrak{R}^t$ on $\mathbb{F}(M)$ as follows:

$$(A, B) \in \mathfrak{R}^t \quad \Leftrightarrow \quad U(\mu_A; t) \cap L(\gamma_A; t) = U(\mu_B; t) \cap L(\gamma_B; t)\,,$$

for all $A = (\mu_A; \gamma_A), B = (\mu_B, \gamma_B) \in \mathbb{F}(M)$. It is easily seen that $\mathfrak{R}^t$ is an equivalence relation on $\mathbb{F}(M)$.

Theorem 3.11. *For any $t \in (0,1)$, the map $\Phi_t : \mathbb{F}(M) \to \mathbb{I}(M) \cup \{\emptyset\}$ defined by $\Phi_t(A) = f_t(A) \cap g_t(A)$ for all $A = (\mu_A, \gamma_A) \in \mathbb{F}(M)$ is injective. Furthermore, the quotient set $\mathbb{F}(M)/\mathfrak{R}^t$ is equipotent to $\mathbb{I}(M) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathbb{F}(M)$, we obtain

$$\Phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) \cap L(\mathbf{1}; t) = \emptyset.$$

For any $J \in \mathbb{I}(M)$, there exists $J_\sim = (\chi_J, \bar{\chi}_J) \in \mathbb{F}(M)$ such that

$$\Phi_t(J_\sim) = f_t(J_\sim) \cap g_t(J_\sim) = U(\chi_J; t) \cap L(\bar{\chi}_J; t) = J.$$

Hence Φ is surjective. Define a map $\Phi_t^* : \mathbb{F}(M)/\mathfrak{R}^t \to \mathbb{I}(M) \cup \{\emptyset\}$ by $\Phi_t^*([A]_{\mathfrak{R}^t}) = \Phi_t(A)$ for all $[A]_{\mathfrak{R}^t} \in \mathbb{F}(M)/\mathfrak{R}^t$. Let $[A]_{\mathfrak{R}^t}, [B]_{\mathfrak{R}^t} \in \mathbb{F}(M)/\mathfrak{R}^t$ be such that $\Phi_t^*([A]_{\mathfrak{R}^t}) = \Phi_t^*([B]_{\mathfrak{R}^t})$. Then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is,

$$U(\mu_A; t) \cap L(\gamma_A; t) = U(\mu_B; t) \cap L(\gamma_B; t).$$

This implies that $(A, B) \in \mathfrak{R}^t$, and so $[A]_{\mathfrak{R}^t} = [B]_{\mathfrak{R}^t}$. Therefore Φ_t^* is injective. Now for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathbb{F}(M)$ we have

$$\Phi_t^*\Big([\mathbf{0}_\sim]_{\mathfrak{R}^t}\Big) = \Phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) \cap L(\mathbf{1}; t) = \emptyset.$$

For $J_\sim = (\chi_J, \bar{\chi}_J) \in \mathbb{F}(M)$, we get

$$\Phi_t^*\Big([J_\sim]_{\mathfrak{R}^t}\Big) = \Phi_t(J_\sim) = f_t(J_\sim) \cap g_t(J_\sim) = U(\chi_J; t) \cap L(\bar{\chi}_J; t) = J.$$

Thus Φ_t^* is surjective. This completes the proof. □

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