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LINEAR TRANSFORMATIONS ON POLYNOMIAL MODELS OF TIME SERIES

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Abstract: This paper studies polynomial modeling of time series. It introduces methods of using linear transformations to help construct polynomial models for an arbitary time series. It proves that a time series has linear models if the reduced column-echelon form of the associated matrix is diagonal, in particular, if it is of full rank. An upper bound for the minimum degree of all polynomial models is provided in case the reduced column-echolon form of the associated matrix is not diagonal. If the time series has $m+1$ time steps, then the minimal total degree of its polynomial models is less than or equal to $m+1-\operatorname{rank}(M)$, where M is the associated matrix. As applications of the theorems, examples in various cases are investigeted.

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1. Introduction

One of the applications of time series is in the study of genes in Biology. There are assumed to be n genes in certain environment and researchers want to analyze the changes of those genes and their interactions over time. In somewhat naive and simplified language, a time series with $m+1$ time steps and n genes starts from a row of data $A_1 = (a_{11}, a_{12}, \ldots, a_{1n})$, where a_{1j} is the measurement of the j-th gene at the first time step t_1, taking a real value. At each later time step t_i, researchers conduct a laboratory experiment and record a row of new data $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. After $m+1$ such laboratory experiments, a time series of n states and $m+1$ time steps is produced. One expects to use the data from this time series to find a vector function $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, from $\mathbb{R}^n$ to itself, to best describe the genes and predict their future behavior. The related reverse engineering problem is also of high interest.

In this context the function $\mathbf{f}$ satisfies $\mathbf{f}(A_i) = A_{i+1}$, $1 \leq i \leq m$. The most natural choice of such desired functions would be polynomials for their simplicity in computing. In fact, if the values of data are taken from a finite field $\mathbf{k}$ of positive characteristic, any function from $\mathbf{k}^n$ to $\mathbf{k}^n$ is an n-vector of polynomial functions. There has been some extensive research over finite fields; see [2], [3], [5], [4], etc. However, this paper will start with polynomial models over commutative rings, and later focus on fields.

Let R be a commutative ring with 1 and $R[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over R.

Definition 1.1. A time series S (over a commutative ring R) with n states and $m+1$ time steps is a set of $m+1$ distinct row vectors in R^n, say $S = \{A_1, A_2, \ldots, A_{m+1}\}$. A model of the time series S is a function $\mathbf{f} = (f_1, f_2, \ldots, f_n) : R^n \to R^n$ such that $f(A_i) = A_{i+1}$ for $i = 1, 2, \ldots, m$. If all f_i's are polynomial functions, we call $\mathbf{f}$ a model of polynomial type. Similarly we can define models of linear, quadratic, or cubic types, etc. We denote by $P(S)$ the set of all models of polynomial type of the time series S.

Remark. One of the immediate restrictions is that no two identical rows are allowed. Otherwise such desired functions may not exist. Also, the number of experiments in practice is usually far less than the number of genes considered because of the limited resources or the cost of experiments. Thus throughout the paper, we assume that the vectors in S are distinct and $n \geq m$.

Definition 1.2. We denote a time series S by the following matrix:

$$\mathbf{S} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_m \\ \hline A_{m+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline a_{(m+1)1} & a_{(m+1)2} & \cdots & a_{(m+1)n} \end{bmatrix}.$$

Let

$$\mathrm{M}_S = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{bmatrix} \quad \text{and} \quad \overline{\mathrm{M}}_S = \begin{bmatrix} A_2 \\ A_3 \\ \cdots \\ A_{m+1} \end{bmatrix}$$

and call M_S the matrix associated to S and $\overline{\mathrm{M}}_S$ the target matrix of S.

Furthermore, any function $\mathbf{g} : R^n \to R^n$ induces a function $\tilde{\mathbf{g}} : M_{m\times n}(R) \to M_{m\times n}(R)$ defined by

$$\tilde{\mathbf{g}}(\mathrm{M}_S) = \begin{bmatrix} \mathbf{g}(A_1) \\ \vdots \\ \mathbf{g}(A_m) \end{bmatrix}.$$

With this definition models of time series can be represented in matrix form; that is, $\mathbf{g}$ is a model of S if and only if $\tilde{\mathbf{g}}(\mathrm{M}_S) = \overline{\mathrm{M}}_S$.

Consider the set $I(S)$ of all polynomials in $R[x_1, x_2, \ldots, x_n]$ vanishing on the set S, i.e., $I(S) = \{\, h \in R[x_1, x_2, \ldots, x_n] \bigm| h(A_i) = 0 \text{ for all } A_i \in S\}$. Then $I(S)$ is an ideal of $R[x_1, x_2, \ldots, x_n]$, called *the vanishing ideal*. All polynomial models of S can be expressed in the form $\mathbf{g} + I(S)$, where $\mathbf{g}$ is a particular polynomial model for S. The ideal $I(S) = \cap\, I(\{A_i\})$ is a radical ideal of $R[x_1, x_2, \ldots, x_n]$ and is finitely generated, with each ideal $I(\{A_i\}) = (x_i - a_{i1}, x_i - a_{i2}, \ldots, x_i - a_{in})$ being maximal.

One of the major concerns in the study of time series is how to find an optimized polynomial model for any given time series. From the context we know that finding a certain polynomial model $\mathbf{f}$ is equivalent to choosing an appropriate element in $I(S)$ combined with a "favorable" $\mathbf{g}$. Thus finding an appropriate generating set of the ideal $I(S)$ plays an important role in the search for optimized polynomial models.

There are different measurements of being a "good" model for a time series. For example, if we are concerned with computational cost or number of variables in the functions, then a good model is computationally economical to find or involves the least number of variables. One of the optimizations we are interested in is to minimize the number of variables occurring in the polynomials. Another is to minimize the total degrees of the polynomial models.

While it is trivial that there will be no maximum for the total degree of a polynomial model of a time series because there is no upper bound on the degree of polynomials in the ideal $I(S)$, the search for the minimal total degree which a polynomial model may reach remains an active topic of study. In practice with real data the actual computation, using *Macaulay* or *CoCoa* to find a Gröbner basis of the ideal $I(S)$, frequently gives generating sets with most terms linear or quadratic, although it is known that the complexity of Gröbner basis calculation is doubly exponential (Laubenbacher et al [6]). It is natural to ask questions such as, "What time series may/may not have models of linear type or quadratic type?" "Are there models which involve only certain given variables?" Recently, research has been done regarding these questions. For example, equivalent conditions for a model to be of linear type, in terms of the number of variables occurring in the polynomials involved, has been done in [2]. Methods to construct such models are developed in [3].

Definition 1.3. Let S be a time series and $\mathbf{f} = (f_1, \ldots, f_n)$ be a model of S of polynomial type. We define the degree of $\mathbf{f}$ and the degree of S, denoted by $\deg(\mathbf{f})$ and $\mu(S)$ respectively, as follows:

$$\deg(\mathbf{f}) = \max\{\text{total degrees of } f_i \;\big|\; i = 1, 2, \ldots, n\};$$

$$\mu(S) = \min\{\deg(\mathbf{f}) \;\big|\; \mathbf{f} \in P(S)\}.$$

In this paper, our main focus is on the optimization of polynomial models in terms of searching for minimal total degrees. In Section 2 we introduce invertible linear transformations to simplify the construction process to produce polynomial models. The linear transformation reduction greatly simplifies the algorithms and enables us to show, from a linear algebraic point of view, that many time series have linear models. We prove that a time series has linear models if the reduced column-echelon form of the associated matrix is diagonal, in particular, if the associated matrix is of full rank. We further illustrate a constructive method to produce linear models for an arbitary time series, after the column reduction. Section 3 contains one of the main theorems of this paper (Theorem 3.4) which gives an upper bound for the minimal degree of a time series. We prove that if $m + 1$ is the number of the time steps for a time series S and M_S is the associated matrix, then the minimal degree $\mu(S)$ of the polynomial models satisfies $\mu(S) \leq m-$ rank$(M_S) + 1$. We also develop an explicit method of constructing polynomial models after the linear transformation reduction. In Section 4, as applications of our theorems we investigate examples of time series in various cases and explicitly determine the bounds for their minimal degrees.

2. Invertible Transformations on Time Series

One of problems in the study of time series is how to search for polynomial models with minimal degrees or the least occurrence of variables. Let S be a time series with n states and $m+1$ time steps. We are especially interested in finding models of certain types, such as models whose degrees are equal to the minimal degree of S or models that involve certain variables of interest. Assume we have in hand a model $\mathbf{f}$ of polynomial type. If we are not satisfied with this one, we can use it to search for others by choosing a Gröbner basis of the vanishing ideal $I(S)$ with respect to a certain term order [1]. Notice that $R[x_1, x_2, \dots, x_n]/I(S) \cong R^m$, the m-dimensional free R-module. Thus carefully selecting the R-basis in the quotient module $R[x_1, x_2, \dots, x_n]/I(S)$ would be another approach to find good models. But these approaches depend on the term order selected and the initial model used. They may require a high level of computation.

In this section, we give a different approach by applying invertible linear transformations of the m-dimensional free R-module to reduce the matrix associated to S into a simpler form in order to obtain a desired polynomial model. Our main focus is to find the minimal degree of S or give an upper bound for it.

Let $\mathbf{T} : R^n \to R^n$ be an invertible linear transformation. Under the standard basis, $\mathbf{T}$ is represented by an $(n \times n)$ invertible matrix $\mathrm{T} = (t_{ij})$ with entries in R. For any time series $S = \{A_1, A_2, \dots, A_{m+1}\}$, $\mathbf{T}$ gives a new time series

$$S' = \{\mathbf{T}(A_1), \mathbf{T}(A_2), \dots, \mathbf{T}(A_{m+1})\} = \{A_1\mathrm{T}, A_2\mathrm{T}, \dots, A_{m+1}\mathrm{T}\}.$$

Obviously, the matrix associated to the new time series is $\mathrm{M}_{S'} = \mathrm{M}_S\mathrm{T}$ and the new target matrix is $\overline{\mathrm{M}}_{S'} = \overline{\mathrm{M}}_S\mathrm{T}$. Further, if $\mathbf{f} = (f_1, f_2, \dots, f_n) : R[\underline{x}]^n \to R[\underline{x}]^n$ is an n-tuple of polynomials in $R[\underline{x}]$, where $\underline{x} = (x_1, x_2, \dots, x_n)$, then $\mathbf{T} \circ \mathbf{f} \circ \mathbf{T}^{-1} : R[\underline{x}]^n \to R[\underline{x}]^n$ is also an n-tuple of polynomials in $R[\underline{x}]$, defined by

$$\mathbf{T} \circ \mathbf{f} \circ \mathbf{T}^{-1}(\underline{x}) = \left(f_1(\underline{x}\mathrm{T}^{-1}), \dots, f_n(\underline{x}\mathrm{T}^{-1})\right)\mathrm{T}.$$

In particular, we have the following theorem.

Theorem 2.1. *Let S, $\mathbf{T}$, and S' be as above. Then the invertible linear transformation $\mathbf{T}$ induces a one-to-one correspondence between the sets $P(S)$ and $P(S')$ of polynomial models which preserves degrees. Moreover, if $\mathbf{f}$ is a polynomial model of the time series S, then $\mathbf{g} = \mathbf{T} \circ \mathbf{f} \circ \mathbf{T}^{-1}$ is a polynomial model of the time series S' with* $\deg(\mathbf{f}) = \deg(\mathbf{g})$*. In particular,* $\mu(S) = \mu(S')$.

Proof. For every $\mathbf{f} \in P(S)$, $\mathbf{f}(\mathrm{M}_S) = \overline{\mathrm{M}}_S$. Then

$$\begin{aligned} \mathbf{g}(\mathrm{M}_{S'}) &= \mathbf{g}(\mathrm{M}_S T) = (\mathbf{T} \circ \mathbf{f} \circ \mathbf{T}^{-1})(M_S T) \\ &= \mathbf{T}\left(\mathbf{f}(\mathrm{M}_S T T^{-1})\right) = (\mathbf{f}(\mathrm{M}_S))T = \overline{\mathrm{M}}_S T = \overline{\mathrm{M}}_{S'}. \end{aligned}$$

This shows that $\mathbf{g} \in P(S')$. It is straightforward to check that the correspondence sending $\mathbf{f}$ to $\mathbf{g}$ is a one-to-one correspondence from $P(S)$ to $P(S')$. Notice that all operations here are linear and T is an invertible matrix over R, so $\deg(\mathbf{g}) \leq \deg(\mathbf{f})$ since $\mathbf{g} = \mathbf{T} \circ \mathbf{f} \circ \mathbf{T}^{-1}$. On the other hand, $\deg(\mathbf{f}) \leq \deg(\mathbf{g})$ by $\mathbf{f} = \mathbf{T}^{-1} \circ \mathbf{g} \circ \mathbf{T}$. Therefore $\deg(\mathbf{f}) = \deg(\mathbf{g})$ and thus $\mu(S) = \mu(S')$. □

Recall that over a field any matrix M can be transformed to a reduced column-echelon form by multiplying M by an invertible matrix on the right. The significance of the above theorem is that finding models of a time series associated with a given matrix can be reduced to finding models (with the same degrees) of a time series with the associated matrix in its reduced column-echelon form, which is much simpler. This gives the following immediate corollary.

Corollary 2.2. *Over a field, to compute polynomial models of a time series, it suffices to assume that the $m \times n$-matrix ($m \leq n$) associated to the time series is in the reduced column-echelon form*

$$\begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 & \cdots & 0 \\ & \cdots & & \ddots & 0 & 0 & \cdots & 0 \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mm} & 0 & \cdots & 0 \end{bmatrix}. \tag{1}$$

□

Corollary 2.3. *Assume* M_S *is the ($m \times n$) nonzero matrix associated to a time series S (with distinct points) and its reduced column-echelon form is* $[\mathrm{B}\ \mathbf{0}]$*, where* B *a square diagonal matrix. Then the rank of* M_S *is at least $m-1$.*

Proof. This is an immediate consequence of Corollary 2.2. Notice that all rows in B must be distinct; thus, B has at most one zero diagonal element. □

If the time series in consideration is from a real world application, such as from gene regulatory data, then the number of columns of the matrix M_S (i.e., the number of samples) is much bigger than the number of rows (i.e., the number of experiments), because of the cost of conducting the experiments. So in the overwhelming majority of the cases in practice the matrix M_S will be of full rank. The next theorem shows that a time series S has linear models provided the associated matrix M_S is of full rank. The proof heavily uses the linear transformation method stated in Theorem 2.1 and constructively produces a linear model.

Theorem 2.4. *Assume a time series S has more than two points. If the reduced column-echelon form of the associated matrix M_S is diagonal, then S has linear models; that is, $\mu(S) = 1$. In particular, S has linear models when M_S is of full rank.*

Proof. Assume the last row in the target matrix $\overline{\mathrm{M}}_S$ is $(c_1, c_2, \ldots, c_n)$; namely, $\overline{\mathrm{M}}_S$ has the form

$$\left[\begin{array}{ccccccccc} b_{11} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 & \cdots & 0 \\ & \cdots & & \ddots & 0 & 0 & \cdots & 0 \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mm} & 0 & \cdots & 0 \\ \hline c_1 & c_2 & c_3 & \cdots & c_m & c_{m+1} & \cdots & c_n \end{array}\right].$$

From Corollary 2.3, the rank of M_S is either m or $m-1$. If the rank is m (full rank), then from Corollary 2.2 we may assume M_S has the form $\mathrm{M}_S = [\mathrm{I}_m\ \mathbf{0}]$, where I_m stands for the $(m \times m)$ identity matrix. Obviously, $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ is a linear model of S, where

$$g_j = \begin{cases} c_j x_m & \text{if} \quad j = 1 \quad \text{or} \quad j > m; \\ x_{j-1} + c_j x_m & \text{if} \quad 1 < j \leq m. \end{cases}$$

From Theorem 2.1 the original time series then has a linear model.

If the rank of the associated matrix is $m-1$, without loss of generality we may assume $\mathrm{M}_S = [\mathrm{B}\ \mathbf{0}]$, where $\mathrm{B} = \mathrm{diag}\ (b_1, b_2, \ldots, b_m)$ and we may assume each $b_i = 0$ or 1. Also, B has exactly one zero row. Let the k-th row be the zero row. We have the following two cases.

Case 1. $k < m$. A linear model $\mathbf{g} = (g_1, g_2, \dots, g_n)$ is given below:

$$g_j = \begin{cases} c_j x_m & \text{for} \quad j = 1, k \quad \text{or} \quad j \geq m\,, \\ \left(1 - \sum_{i=1}^{m} x_i\right) + c_j x_m & \text{for} \quad j = k+1\,, \\ x_{j-1} + c_j x_m & \text{for} \quad 1 < j < m, \quad i \neq k, k+1\,. \end{cases}$$

Case 2. $k = m$. A linear model $\mathbf{g} = (g_1, g_2, \dots, g_n)$ is given below:

$$g_j = \begin{cases} c_j \left(1 - \sum_{i=1}^{m} x_m\right) & \text{for} \quad j = 1 \quad \text{or} \quad j \geq m\,, \\ x_{j-1} + c_j \left(1 - \sum_{i=1}^{m} x_m\right) & \text{for} \quad 1 < j < m. \end{cases}$$

One can check that in both cases $\mathbf{g}(M_S) = \overline{\mathrm{M}}_S$. □

3. Minimal Degrees of Polynomial Models

From Section 2, we know that a time series S has linear models, that is, $\mu(S) = 1$, if M_S is of full rank. In this section we look at more general cases in which the associated matrix may not have diagonal reduced column-echelon form. We give an estimate of the minimal total degree of the polynomial models. We also give some examples of the time series to show that our estimation is the best one.

Throughout this section we assume that the associated matrix M_S is of rank l, where $1 \leq l \leq m$, and its reduced column-echelon form is $[\mathbf{B}\ \mathbf{0}]$. Suppose $r_1, \dots, r_m$ are the rows of $\mathbf{B}$, among which $r_{i_1}, r_{i_2}, \dots, r_{i_l}$ are the rows with a leading 1. Notice that 1 is the only non-zero entry in each of those rows. Let $r_m = [b_1, \dots, b_n]$. Before we prove our main theorem, we need the following lemmas. The first lemma deals with the rows with a leading 1 while the second lemma deals with the other rows.

Lemma 3.1. *There exists a polynomial* $\alpha(\underline{x}) = \alpha(x_1, \dots, x_n)$ *of total degree at most* 2 *such that* $\alpha(R_{i_1}) = \alpha(R_{i_2}) = \cdots = \alpha(R_{i_l}) = 0$ *but* $\alpha(R_m) = 1$.

Proof. One can easily verify that the function $\alpha(\underline{x})$ defined below satisfies the requirements.

Case 1. $i_l = m$. Define $\alpha(x_1, \dots, x_n) = x_l$.

Case 2. $i_l < m$ and $b_1 + \cdots + b_l \neq 1$. Define

$$\alpha(x_1, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_l - 1}{b_1 + b_2 + \cdots + b_l - 1}.$$

Case 3. $i_l < m$ and $b_1 + \cdots + b_l = 1$. In this case the m-th row is not a zero row. Then this row must have at least two nonzero entries; otherwise it will be identical to a row with a leading 1. Furthermore, at least one of the nonzero entries is not 1, say, $0 \neq b_{mk} \neq 1$ $(1 \leq k \leq l)$. We define

$$\alpha(x_1, \ldots, x_n) = \frac{\left((\sum_{i=1}^{l} x_i) - x_k - 1\right)(1 - x_k)}{b_{mk}}.$$

This α satisfies all the requirements. ☐

Lemma 3.2. *When $l < m$, there exists a polynomial $\alpha(\underline{x}) = \alpha(x_1, \ldots, x_n)$ of total degree $m - l - 1$ such that $\alpha(r_m) = 1$ and $\alpha(r_i) = 0$ for every row r_i of $\mathbf{B}$ which does not contain a leading 1 and $i < m$.*

Proof. We compare each row r_i as above with the last row r_m. Since no two rows are identical, for each $r_i = [b_{i1}, b_{i2}, \ldots, b_{in}]$, there exists $1 \leq j_i \leq l$ with $b_{ij_i} \neq b_{mj_i}$. Noting that there are at most $m - l - 1$ such rows r_i, one can easily verify that the following definition satisfies the requirements.

$$\alpha(\underline{x}) = \prod_{i \notin \{i_1, i_2, \ldots, i_l\}}^{m-1} \frac{x_{j_i} - b_{ij_i}}{b_{mj_i} - b_{ij_i}}.$$ ☐

Lemma 3.3. *Assume $l < m$. There exists a polynomial $\alpha(\underline{x}) = \alpha(x_1, \ldots, x_n)$ of total degree $m - l + 1$ such that $\alpha(r_i) = 0$ for $i = 1, 2, \ldots, m-1$ and $\alpha(r_m) = 1$.*

Proof. Multiplying two appropriate polynomials from Lemma 3.1 and Lemma 3.2 we have the desired polynomial. ☐

Theorem 3.4. *Let S be any time series with $m+1$ points in $\mathbb{R}^n$. Without loss of generality, we assume that the associated matrix $M_S = [\mathbf{B}\,\mathbf{0}]$ is in the reduced column-echelon form with rank l. Then there exists a polynomial model $\mathbf{f} = (f_1, \cdots, f_n)$ for S such that $\deg \mathbf{f} \leq m - l + 1$. Hence the minimal degree $\mu(S) \leq m - l + 1$.*

Proof. Let M_S be in its reduced column-echelon form $M_S = [\mathbf{B}\,\mathbf{0}]$. Let the last point in S be $[c_1, c_2, \ldots, c_n]$. When $l = m$, the theorem is true since S has a linear model by Theorem 2.4. We now assume $l < m$ and prove the theorem by mathematical induction on m.

When $m = 1$, the theorem is clear.

Assume the theorem is true for all time series of $m-1$ points. We consider the matrix $\mathbf{B}$ with m rows, which is in the reduced column-echelon form and associated to the time series S with m points. Let $\mathbf{B}'$ be the first $m-1$ rows of $\mathbf{B}$. We view $\mathbf{B}'$ as the matrix associated to the time series S' consisting of the first $m-1$ points of S and the m-th row of $\mathbf{B}$ as the last row in the target matrix of S'. Notice that

$$\text{rank}(\mathbf{B}) - 1 \leq \text{rank}(\mathbf{B}') \leq \text{rank}(\mathbf{B}).$$

By the induction hypothesis, there exists a polynomial model $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ with

$$\deg(\mathbf{g}) \leq (m-1) - \text{rank}(\mathbf{B}') + 1 \leq m - 1 - (l-1) + 1 = m - l + 1.$$

Note that $\mathbf{g}(r_i) = r_{i+1}$, for $i = 1, 2, \ldots, m-1$. We extend $\mathbf{g}$ to a model $\mathbf{f}$ which, in addition, sends the last row of $\mathbf{B}$ to $(c_1, c_2, \ldots, c_n)$. By Lemma 3.3, for every j, $1 \leq j \leq l$, we have $\alpha_j(\underline{x})$ such that $\alpha_j(r_i) = r_{i+1}$ for $i = 1, 2, \ldots, m-1$ and $\alpha_j(r_m) = 1$. The total degree of α is at most $m - l + 1$. We define

$$f_j(\underline{x}) = g_j(\underline{x}) + \alpha_j(\underline{x})(c_j - g_j(r_m)) \quad \text{for } 1 \leq j \leq l\,,$$

$$f_j(\underline{x}) = g_j(\underline{x}) \quad \text{for } j > l\,,$$

and

$$\mathbf{f} = (f_1, f_2, \ldots, f_n).$$

One can check that the above defined $\mathbf{f}$ is a model for the time series S and $\deg(\mathbf{f}) \leq m - l + 1$. □

4. Examples in Special Cases

In this section we focus on polynomial models of a time series with low degrees, especially those having the minimal degree. We provide several examples covering different cases, which involve time series whose associated matrix is of either full rank or not. All time series considered in this section are over a field.

Example 4.1. A time series S consists of five points: $P_1 = [1, 1, 0, -1, 1]$, $P_2 = [2, -1, 0, 3, 2], P_3 = [0, 1, 3, 2, 1]$, and $P_4 = [23, -1, -11, 14, 8]$. We show that $\mu(S) = 1$ by finding a linear model for S.

The matrix form of S is:

$$\mathbf{S} = \left[\begin{array}{ccccc} 1 & 1 & 0 & -1 & 1 \\ 2 & -1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ \hline 23 & -1 & -11 & 14 & 8 \end{array}\right] \quad \text{with} \quad M_S = \left[\begin{array}{ccccc} 1 & 1 & 0 & -1 & 1 \\ 2 & -1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 2 & 1 \end{array}\right].$$

The associated matrix M_S is of full rank.

By applying a linear transformation $\mathbf{T}$, represented by an invertible matrix T, we obtain a new time series S' whose associated matrix is in its reduced column-echelon form. Here

$$S' = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 3 & -1 & 4 & 2 \end{array}\right]$$

and

$$\mathrm{T} = \frac{1}{202}\left[\begin{array}{ccccc} 28 & -27 & 38 & 57 & -12 \\ 18 & -39 & 10 & 15 & 50 \\ 12 & -26 & 74 & 10 & -34 \\ -70 & 17 & 6 & 9 & 30 \\ 86 & 83 & -42 & -63 & -8 \end{array}\right].$$

By Theorem 2.4, we can easily construct a linear model $\mathbf{g}$ for S', which is shown below:

$$\mathbf{g} = \left[\ x_3, x_1 + 3x_3, x_2 - x_3, 4x_3, 2x_3\ \right].$$

We then use the methods provided by Theorem 2.1 to obtain a linear model $\mathbf{f}$ for S:

$$\mathbf{f} = \mathbf{T}^{-1} \circ \mathbf{g} \circ \mathbf{T} = (f_1, f_2, f_3, f_4, f_5),$$

where

$$\begin{cases} f_1 = \frac{465}{101}x_1 + \frac{133}{101}x_2 + \frac{863}{101}x_3 - \frac{1}{101}x_4 - \frac{397}{101}x_5\,, \\ f_2 = -\frac{93}{202}x_1 - \frac{67}{202}x_2 + \frac{56}{101}x_3 + \frac{81}{202}x_4 + \frac{39}{202}x_5\,, \\ f_3 = -\frac{499}{202}x_1 - \frac{227}{202}x_2 - \frac{446}{101}x_3 - \frac{15}{202}x_4 + \frac{711}{202}x_5\,, \\ f_4 = \frac{281}{101}x_1 + \frac{58}{101}x_2 + \frac{510}{101}x_3 - \frac{46}{101}x_4 - \frac{82}{101}x_5\,, \\ f_5 = \frac{333}{202}x_1 + \frac{77}{202}x_2 + \frac{295}{101}x_3 - \frac{75}{202}x_4 - \frac{81}{202}x_5. \end{cases}$$

One can easily check that $\mathbf{f}(P_i) = P_{i+1}$ for $i = 1, 2, 3$.

Example 4.2. The associated matrix of a time series S is given below:

$$\left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ b_1 & 0 & \cdots & 0 \\ \hline c_1 & c_2 & \cdots & c_n \end{array}\right].$$

We may assume $b_1 \neq 1$ because of the assumption that M_S does not have identical rows. A linear model for S of degree 1 is given by $\mathbf{f} = (f_1, \cdots, f_n)$, where

$$f_1 = b_1 x_1 + \frac{c_1 - b_1^2}{b_1 - 1}(x_1 - 1) \quad \text{and} \quad f_j = \frac{x_1 - 1}{b_1 - 1} c_j \quad \text{for} \quad j > 1. \tag{2}$$

Thus $\mu(S) = 1$ in this case.

Example 4.3. The associated matrix of a time series S is shown below:

$$\left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ b_1 & 0 & \cdots & 0 \\ b_2 & 0 & \cdots & 0 \\ \hline c_1 & c_2 & \cdots & c_n \end{array}\right].$$

Again we may assume $b_2 \neq b_1 \neq 1$. Assume a linear model $\mathbf{f} = (f_1, \cdots, f_n)$ exists, say,

$$f_1(\underline{x}) = a_1 x_1 + v(x_2, \ldots, x_n),$$

where a_1 is a scalar in the ground field and v is a polynomial involving only the variables $x_2, \ldots, x_n$. We have

$$\left[\begin{array}{lcll} f_1(1, 0, \cdots, 0) & = & a_1 + v(0, \cdots, 0) & = b_1, \\ f_1(b_1, 0, \cdots, 0) & = & a_1 b_1 + v(0, \cdots, 0) & = b_2, \\ f_1(b_2, 0, \cdots, 0) & = & a_1 b_2 + v(0, \cdots, 0) & = c_1. \end{array}\right.$$

This implies

$$\left| \begin{array}{ccc} 1 & 1 & b_1 \\ b_1 & 1 & b_2 \\ b_2 & 1 & c_1 \end{array} \right| = 0 \Longleftrightarrow \frac{c_1 - b_2}{b_2 - b_1} = \frac{b_2 - b_1}{b_1 - 1}. \tag{3}$$

If (3) is true, one can check that $\mathbf{f} = (f_1, f_2, f_3)$ gives a linear model, where

$$f_1 = \left(b_1 + \frac{c_1 - b_1^2}{b_1 - 1}\right) x_1 + \frac{b_1^2 - c_1}{b_1 - 1} \quad \text{and} \quad f_j = \frac{x_1 - 1}{b_1 - 1} c_j \quad \text{for} \quad j = 2, 3.$$

On the other hand, if (3) is false, there is no linear model. However, a quadratic model exists. Let $g_1 = b_1x_1 + \frac{b_2-b_1^2}{b_1-1}(x_1-1)$. We construct the following functions:

$$f_1 = (c_1 - g_1(b_2, 0, \ldots, 0)) \cdot \frac{x_1 - 1}{b_2 - 1} \cdot \frac{x_1 - b_1}{b_2 - b_1} + g_1(\underline{x})$$

and

$$f_j = \frac{x_1 - 1}{b_2 - 1} \cdot \frac{x_1 - b_1}{b_2 - b_1} \cdot c_j \qquad \text{for} \quad j = 2 \text{ or } 3.$$

Then $\mathbf{f} = (f_1, f_2, f_3)$ is a quadratic model. In conclusion, $\mu(S) = 1$ if and only if (4) is true, and $\mu(S) = 2$ otherwise.

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