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# WEIERSTRASS $n$-PLES ON SMOOTH CURVES

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**Abstract:** Here we give a characteristic free definition of gap sequences and weights for Weierstrass $n$-ples of linear systems on smooth curves and an existence theorem for them (in large characteristic).

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**Key Words:** Weierstrass $n$-ple, Weierstrass point, ramification point, gap sequence

## 1. Weierstrass $n$-Ples on Smooth Curves

Let $X$ be a smooth and connected curve of genus $g \geq 0$ defined over the algebraically closed field $\mathbb{K}$, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, L)$ a linear subspace. Set $r := \dim(V) - 1$. Fix an integer $m > 0$, integers $a_i \geq 0$ such that $\sum_{i=1}^{n} a_i = r + 1$ and $n$ distinct points $P_i \in X$, $1 \leq i \leq n$. Fix any permutation $\sigma$ of the set $\{1, \ldots, n\}$. Set $a_0 := 0$. Our definitions of weights and gap sequences will depend from the choice of $\sigma$. For each integer $i \in \{1, \ldots, n\}$ we define a map $\beta_i : \{\sum_{j=1}^{i-1} a_j, \ldots, \sum_{j=1}^{i} a_j - 1\} \to \mathbb{N}$ (or $\beta_{i;P_1,\ldots,P_n}$) in the following way. Let

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$\beta_1(0)$ be the first integer $x$ such that $\dim(V(-(x+1)P_{\sigma(1)})) < r+1$. Now take an integer $t \in \{1, \dots, a_1 - 1\}$ and assume defined the integer $\beta_1(t-1)$. Let $\beta_1(t)$ be the first integer $x > \beta_1(t-1)$ such that $\dim(V(-(x+1)P_{\sigma(1)})) < \dim(V(-(\beta_1(t-1)+1)P_{\sigma(1)}))$. Now fix $i \in \{2, \dots, m\}$ and assume defined the functions $\beta_j$ for all $j < i$. Take an integer $t \in \{\sum_{j=1}^{i-1} a_j, \dots, \sum_{j=1}^{i} a_j\}$. Let $\beta_i(0)$ be the minimal integer $x \geq \sum_{j=1}^{i-1} a_j$ such that $\dim(V(-(x+1)P_{\sigma(i)} - \sum_{j^1}^{i-1} \beta_j(a_j)P_{\sigma(j)})) < \dim(V(-\sum_{j^1}^{i-1} \beta_j(a_j)P_{\sigma(j)}))$. We will say that $\beta_i$ is the $i$-th gap sequence or $i$-th gap function of $(X, L, V, P_1, \dots, P_n, a_1, \dots, a_n, \sigma)$. Now assume that $(P_1, \dots, P_n)$ is general in $X^n$. The corresponding gap sequence $\beta_i$ will be called the generic gap sequences of $(X, L, V, \sigma)$ and often denoted with $\beta_{i,gen}$ or with $\psi_i$. If $\beta_{i;P_1,\dots,P_n} \neq \psi_i$ for some $i$, then we will say that $(P_1, \dots, P_n)$ is a Weierstrass $n$-ple for $(X, L, V, a_1, \dots, a_m, \sigma)$. The gap weight for $(X, L, V, a_1, \dots, a_m, \sigma)$ may be defined as in the classical case (see [1] for the canonical linear system). For each integer $i$, $1 \leq i \leq n$ it is natural to define the $i$-th weight of $(X, L, V, a_1, \dots, a_m, \sigma)$ as the weight in the sense of [3] at $P_{\sigma(i)}$ of the linear system $|V(-\sum_{j=1}^{i-1} a_j P_j)|$. A more refined weight is obtained using the function $\beta_j$ instead of the integer $b_j$, $1 \leq j \leq i-1$.

We fix integers $n > 0$, integers $b_i > 0$ and $n$ distinct points $P_1, \dots, P_n$. Any ramification point $Q$ of the linear system $|V(-b_1P_1 - \cdots - b_nP_n)$ will be called a weak Weiertrass point of $(V, P_1, \dots, P_n, b_1, \dots, b_n)$. We will say that $Q$ is a proper Weierstrass point of $(V, P_1, \dots, P_n, b_1, \dots, b_n)$ if it is a weak Weierstrass point of $(V, P_1, \dots, P_n, b_1, \dots, b_n)$ and $Q \notin \{P_1, \dots, P_n\}$. For the omitted case $n = 0$ (i.e. for the classification of all triples $(X, L, V)$ without any Weierstrass point, see [2]).

To study the existence of proper Weierstrass points we need the following well-known result.

**Lemma 1.** *Let $A$ be a connected a smooth curve, $P \in A$, $L \in \mathrm{Pic}^d(A)$ and $V \subseteq H^0(X, L)$ a linear subspace. Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > d$. *Set $g := p_a(X)$ and $r := \dim(V) - 1$. Assume $r \geq 0$. Let $w$ be the weight of $P$ for the linear system $|V|$ Then $w \leq (r+1)(d-r-1)$ and $w = (r+1)(d-r-1)$ if and only if $V = xQ + H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d-x))$ for some integer $x > 0$.*

*Proof.* By our assumption on $\mathrm{char}(\mathbb{K})$ the linear system $|V|$ has classical gap sequence and the weight $w$ of $P$ may be computed using the Hermite invariants of $P$ for $|V|$ ([3], Theorem 15, when $\mathrm{char}(\mathbb{K}) = 0$, its proof when $\mathrm{char}(\mathbb{K}) > d$; see in particular the formula the weight in the middle of [3], p. 63). In particular, if we increase the Hermite invariants of $P$, then we increase $w$. Hence $w \leq w'$, where $w'$ is the weight for a point $Q \in A$ of the linear system $(d-r)Q + W$ with $\dim(V) = r+1$. In this case we have $w' = (r+1)(d-r-1)$.

Since $\deg(L(-(d-r)Q)) = d-r$, if $w = w'$ we also obtain $A \cong \mathbf{P}^1$. Alternatively, use the Brill-Segre Formula ([3], p. 62). $\square$

Lemma 1 and the Brill-Segre Formula (see [3], p. 62) immediately give the following cheap existence theorem for proper Weierstrass points.

**Theorem 1.** *Fix an integer $n \geq 1$ and integers $b_i > 0$, $1 \leq i \leq n$. Let $X$ be a smooth and connected projective curve of genus $g$, $P_1, \dots, P_n$ distinct points of $X$, $L \in \mathrm{Pic}^d(X)$ and $V \subseteq H^0(X, L)$ a non-zero linear subspace. Set $\eta := \dim(V(-\sum_{i=1}^n b_i P_i))$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > d - b_1 - \dots - b_n$. Assume $(g-1)\eta > (n-1)(d - \sum_{i=1}^n b_i)$. Then there is $Q \in X \backslash \{P_1, \dots, P_n\}$ which is a Weierstrass point of $V(-\sum_{i=1}^n b_i P_i)$.*

**Remark 1.** The inequality $(g-1)\eta > (n-1)(d - \sum_{i=1}^n b_i)$ in the statement of Theorem 1 is always satisfied if $\eta \geq 2$ (i.e. if the problem makes any sense) and $(n-1)(d - b_1 - \dots - b_n) < 2g - 2$.

**Example 1.** Assume $p := \mathrm{char}(\mathbb{K}) \geq 3$. Let $X$ be a supersingular elliptic curve and $O \in X$. Use $O$ to define the group structure on the points of $X$. Set $L := (p+1)O$ and $V := H^0(X, L)$. Hence $L$ and $L(-O)$ are very ample. The linearly normal embedding associated to $L(O)$ has $O$ as its unique ramification point. Taking $n = 1$ and $b_1 = 1$ we obtain that the restriction on $\mathrm{char}(\mathbb{K})$ in the statement of Theorem 1 is sharp.

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