International Journal of Pure and Applied Mathematics
Volume 17 No. 2 2004, 169-172

ON THE EXISTENCE OF SMOOTH PSEUDOWEIERSTRASS POINTS ON A NON-GORENSTEIN CURVE

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we prove an existence theorem for pseudoweierstrass points on non-Gorenstein integral projective curves.

AMS Subject Classification: 14H20, 14H55
Key Words: Weierstrass point, pseudoweierstrass point, singular curve, canonical model of a singular curve, non-Gorenstein curve

1. Smooth Pseudoweierstrass Points

Let X be an integral projective curve of with arithmetic genus $g := p_a(X) \geq 3$ and $f : Y \to X$ its normalization. In [1] we introduced the following definition of psudoweiertrass point when X is not Gorenstein, i.e. ω_X s not locally free. We have $h^0(X, \omega_X) = g$. Set $\omega := f^*(\omega_X)/\mathrm{Tors}(f^*(\omega_X)) \in \mathrm{Pic}(Y)$. Since f is an isomorphism outside finitely many points, the natural map $f^*(H^0(X, \omega_X)) \to H^0(Y, \omega)$ is injective and hence its image, Λ, has dimension g. Λ spans ω (see [6]) and hence it induces a morphism $\alpha : Y \to \mathbf{P}^{g-1}$. Set $Z := \mathrm{Im}(\alpha)$ and call $\beta : Y \to Z$ the surjective morphism induced by α. Thus $Z \subset \mathbf{P}^{g-1}$ is an

Received: July 28, 2004

integral non-degenerate curve. For more on this set-up, see [6], or [7], or [5], §1.2. From now on we assume that X is not hyperelliptic. This implies that α is birational onto its image (see [6], Theorem 17) and that f factors through β, i.e. there is a morphism $\psi : Z \to X$ such that $f = \psi \circ \beta$ (see [6], Theorem 17, or [5], Theorem 1.3).

Definition 1. A point $P \in Y$ will be called a pseudoweierstrass point of Y if it is a ramification point for the linear system $|V| := \mathbf{P}(f^*(H^0(X, \omega_X)^*)$ on X.

In [3] there is a characteric free classification of all linear systems on smooth curves without any ramification point. Using it (or rather its classical characteristic zero version) we classified in characteristic zero all non-Gorenstein curves without psedoweierstrass points (see [2], Theorem 1): $Y \cong \mathbf{P}^1$, $Y \cong Z$ and Λ is the complete linear system $|\mathcal{O}_{\mathbf{P}^1}(g)|$; in this case we also described the singularities of X. Furthermore, in [2], Example 2, we proved in positive characteristic of an integral curve X such that every pseudoweiertrass point is mapped to a singular point of X, while $Y \ncong \mathbf{P}^1$; in this example Y is a supersingular elliptic curve. The aim of this short note is to prove in one important case the existence of a pseudoweierstrass point $P \in Y$ such that $f(P) \notin \mathrm{Sing}(X)$; any such point will be called a smooth pseudoweierstrass point of X, because P corresponds to a unique point, $f(P)$, of X and $f(P) \in X_{reg}$.

Theorem 1. *Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > 2g - 4$. *Let X be an integral non-Gorenstein projective curve with $g := p_a(X) \geq 3$. Assume that X has at least one pseudoweierstrass point, i.e. assume that Rosenlicht's canonical model of X is not a rational normal curve. Assume X not hyperelliptic,* $\sharp(\mathrm{Sing}(X)) = 1$, *$X$ unibranch at its unique singular point. Then X has a smooth pseudoweierstrass point.*

Notation 1. Let A be a connected a smooth curve, Z a closed subscheme of A, $L \in \mathrm{Pic}(A)$ and $V \subseteq H^0(X, L)$ a linear subspace. Set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes L)$ and $Z + V \subseteq H^0(X, L(Z))$. Hence $Z + V$ is a vector space, $\dim(Z + V) = \dim(V)$ and Z is contained in the base locus of $Z + V$.

Proposition 1. *Let A be a connected a smooth curve, $P \in A$, $L \in \mathrm{Pic}^d(A)$ and $V \subseteq H^0(X, L)$ a linear subspace. Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > d$. *Set $n := \dim(V) - 1$. Assume $n \geq 0$. Let w be the weight of P for the linear system $|V|$ Then $w \leq (n + 1)(d - n)$ and $w = (n + 1)(d - n)$ if and only if $V = xQ + H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d - x))$ for some integer $x > 0$.*

Proof. By our assumption on $\mathrm{char}(\mathbb{K})$ the linear system $|V|$ has classical gap sequence and the weight w of P may be computed using the Hermite invariants of P for $|V|$ (see [4], Theorem 15, when $\mathrm{char}(\mathbb{K}) = 0$, its proof when $\mathrm{char}(\mathbb{K}) > d$) and in the matrix of [4], equation (8), all binomial coefficients are non-zero. In particular, if we increase the Hermite invariants of P, then we increase w. Hence $w \le w'$, where w' is the weight for a point $Q \in A$ of the linear system $(d-n)Q + W$ with $\dim(W) = n+1$. In this case we have $w' = (n+1)(d-n)$. Since $\deg(L(-(d-n)Q)) = d-n$, if $w = w'$ we also obtain $A \cong \mathbf{P}^1$ and $W = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n))$. Alternatively, use the Brill-Segre Formula (see the proof of Proposition 2 below). □

Proposition 2. *Let A be a connected a smooth curve, $L \in \mathrm{Pic}^d(A)$ and $V \subseteq H^0(X, L)$ a linear subspace. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > d$. Assume either $X \ncong \mathbf{P}^1$ or $X = \mathbf{P}^1$ and $V = xQ + H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d-x))$ for some integer x with $0 \le x \le d$. Then V has at least two different ramification points.*

Proof. Set $g := p_a(A)$ and $n := \dim V - 1$. The characteristic free classification (without any assumption $\mathrm{char}(\mathbb{K})$) of all triples (A, L, V) without any ramification point is due to M. Homma (see [3]). By [3] in order to obtain a contradiction we may assume that V has a unique ramification point, P. Let w denote its weight. By our assumption on $\mathrm{char}(\mathbb{K})$ the linear system $|V|$ has classical gap sequence (see [4], Theorem 15, when $\mathrm{char}(\mathbb{K}) = 0$, its proof when $\mathrm{char}(\mathbb{K}) > d$). Hence by the Brill-Segre Formula (see [4], Theorem 9) we have

$$w = n(n+1)(g-1) + d(n+1)\,. \tag{1}$$

Apply Proposition 1. □

Proof of Theorem 1. By assumption $\sharp(f^{-1}(\mathrm{Sing}(X)) = 1$. Apply Proposition 2. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Rosenlicht's canonical model of seminormal curves, *International Journal of Pure and Applied Mathematics*, To Appear.

[2] E. Ballico, Higher order Weiertrass points on non-Gorenstein curves, *International Journal of Pure and Applied Mathematics*, To Appear.

[3] M. Homma, Linear systems on curves with no Weierstrass points, *Bol. Soc. Brasil Mat.*, N.S., **23** (1992), 93-108.

[4] D. Laksov, Wronskians and Plücker formulas for linear systems on curves, *Ann. Scient. École Norm. Sup.*, **17** (1984), 565-579.

[5] R.V. Martins, On trigonal non-Gorenstein curves with zero Maroni invariant, *J. Algebra*, **275**, No. 2 (2004), 453-470.

[6] M. Rosenlicht, Equivalence relations on algebraic curves, *Ann. of Math.*, **56** (1952), 169-191.

[7] K.-O. Stöhr, On the poles of regular differentials of singular curves, *Bol. Soc. Brasil. Mat.*, **24** (1993), 105-135.