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MULTIGRADED POLYNOMIALS AND INTERPOLATION OVER A FINITE FIELD

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Abstract: Here we use the so-called Horace Method to study interpolation problems for multihomogeneous polynomials.

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1. Introduction

Fix a prime p and a p-power q. Let $\mathbb{F}_q$ or $GF(q)$ denote the finite field with q elements. For any integer $n > 0$ let $\mathbb{A}^n(q)$ denote the affine n-dimensional space over $GF(q)$ with a prescribed system of coordinates $x_1, \dots, x_n$. See $\mathbb{A}^n(q)$ as a subset of the n-dimensional projective space $PG(n, q)$ with a prescribed system of homogeneous coordinates $z_0, \dots, z_n$ such that $x_i = z_i/z_0$ for $1 \le i \le n$. Hence $PG(n, q)\backslash\mathbb{A}^n(q)$ is the hyperplane $\{z_0 = 0\}$. Set $\partial_i := \partial/\partial x_i$. Similarly, for any multiindex $\alpha = (a_1, \dots, a_n)$ let ∂_α denote the corresponding differential operator of order $|\alpha| := a_1 + \dots + a_n$, where if $a_i \ge p$ for some i we use the Hasse derivatives (see [2], Section 3, for their elementary properties), not the usual high order partial derivatives. For any integer $d \ge 0$ let $V_{n,d}$ denote the $GF(q)$-vector space of all polynomials of degree at most d in the variables

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$x_1, \ldots, x_n$ and $W_{n,d}$ the $GF(q)$-vector space of all homogeneous polynomials of degree d in the variables $z_0, \ldots, z_n$. Hence $\dim(V_{n,d}) = \dim(W_{n,d}) = \binom{n+d}{n}$. For any $S \subseteq \mathbb{A}^n(q)$ and any $M \subseteq PG(n,q)$ set $V_{n,d}(-S) := \{f \in V_{n,d} : f(P) = 0$ for every $P \in S\}$ and $W_{n,d}(-M) := \{f \in W_{n,d} : f(P) = 0$ for every $P \in M\}$. Hence $\dim(V_{n,d}) \geq \binom{n+d}{n} - \mathrm{card}(S)$ and $\dim(W_{n,d}) \geq \binom{n+d}{n} - \mathrm{card}(M)$.

Remark 1. For all integer d such that $0 \leq d < q$ we have $V_{n,d}(-\mathbb{A}^n(q)) = \{0\}$ and $W_{n,d}(-PG(n,q)) = \{0\}$. Notice that this is false for $d \geq q$ (e.g use $x_1^{d-q}(x_1^q - x_1)$). Hence there are $S \subset \mathbb{A}^n(q)$ and $M \subset PG(n,q)$ such that $\mathrm{card}(S) = \mathrm{card}(M) = \binom{n+d}{n}$ and $V_{n,d}(-S) = W_{n,d}(-M) = \{0\}$. Hence S (resp. M) imposes independent conditions to $V_{n,d}$ (resp. $W_{n,d}$). Hence for all $A \subseteq S$, all $B \subseteq M$, all $S \subseteq E \subseteq \mathbb{A}^n(q)$ and all $M \subseteq F \subseteq PG(n,q)$ we have $\dim(V_{n,d}(-A)) = \binom{n+d}{n} - \mathrm{card}(A)$, $\dim(W_{n,d}(-B)) = \binom{n+d}{n} - \mathrm{card}(B)$, $V_{n,d}(-E) = \{0\}$ and $W_{n,d}(-F) = \{0\}$.

The notions of partial derivatives and of Hasse derivatives depends from the choice of the coordinate system, but the notion of Taylor expansion is coordinate-free and we will recast it in the langauge of schemes and of fat points. Let X be any n-dimensional scheme over a field K and $P \in X_{reg}$ any smooth point of X. Set $0P := \emptyset$. For any integer $m > 0$ let mP denote the closed subscheme of X with $(\mathcal{I}_{P,Z})^m$ as its ideal sheaf. Hence mP is a zero-dimensional subscheme of X, $(mP)_{red} = \{P\}$ and $\mathrm{length}(mP) = \binom{n+m-1}{n}$. Often, mP is called the fat point of X with multiplicity m and with P as its support. If X and P are defined over a subfield L of the base field, then the scheme mP is defined over L. For any germ $f \in \mathcal{O}_{X,P}$ of a regular function near P, its restriction $f_{|mP}$ represent its Taylor's expansion, up to order $m-1$, at P.

This was the set-up of our previous preprint [1]. In particular we proved there a result (see [1], Theorem 2) which, with the notation of the present paper, may be restated in the following form.

Theorem 1. *Fix integers n, d, a, b such that $n > 0$, $a \geq 0$, $b \geq 0$, $d < q$ and $a + b \leq q^n$. Then there exists $A, B \subset \mathbb{A}^n(q)$ with the following properties:*

(a) $\mathrm{card}(A) = a$, $\mathrm{card}(B) = a$ *and* $A \cap B = 0$*;*

(b) for each $P \in B$ there is $i(P) \in \{1, \ldots, n\}$ such that, calling Z the interpolation problem associated to the evaluation at all points of $A \cup B$ and to the partial derivative $\delta_{i(P)}$ at each $P \in B$, the $GF(q)$-vector space of all solutions of Z has dimension $\max\{0, \binom{n+d}{n} - a - 2b\}$.

For our needs here we are forced to write down its proof (see Section 2).

Notice that for every integer $m > 0$, any hyperplane H of $\mathbb{A}^n(q)$ and any $P \in H$ we have $\mathrm{Res}_H(mP) = (m-1)P$ (with the convention $0P := \emptyset$). Just using induction on n, d, m and a hyperplane through P the proof of Theorem 1 gives the following result.

Theorem 2. *Fix integers n, d, a, b, m such that $n > 0$, $a \geq 0$, $b \geq 0$, $0 < m < d < q$ and $a + b < (q^{n+1} - 1)/(q - 1)$. Fix $P \in \mathbb{A}^n(q)$. Then there exists $A, B \subset \mathbb{A}^n(q)$ with the following properties:*

(a) $\mathrm{card}(A) = a$, $\mathrm{card}(B) = a$, $P \notin A \cup B$ *and* $A \cap B = 0$*;*

(b) for each $P \in B$ there is $i(P) \in \{1, \dots, n\}$ such that, calling Z the interpolation problem associated to the union of mP the evaluation at all points of $A \cup B$ and to the partial derivative $\delta_{i(P)}$ at each $P \in B$, the $GF(q)$-vector space of al solutions of Z has dimension $\max\{0, \binom{n+d}{n} - a - 2b - \binom{n+m-1}{n-1}\}$.

Here we want to extends the set-up of [1] to the multigraded case.

Fix integers $s \geq 2$ and $n_i > 0$, $1 \leq i \leq s$, and a field K. $\mathbf{P}_K^{n_1+\cdots+n_s}$ is not the only interesting compactification (or completion) of $\mathbb{A}_K^{n_1+\cdots+n_s}$: $\mathbf{P}_K^{n_1} \times \cdots \times \mathbf{P}_K^{n_s}$ is another interesting compactification and we will show here its connection with multihomogeneous polynomials. If $K = GF(q)$ we will write $\mathbf{P}^{n_1}(q) \times \cdots \times \mathbf{P}^{n_s}_(q)$ intead of $\mathbf{P}_K^{n_1} \times \cdots \times \mathbf{P}_K^{n_s}$. Fix indeterminates $z_{i,j}$, $1 \leq i \leq s$, and $0 \leq j \leq n_s$. Let $R := K[z_{i,j}]$ be the polynomial ring in $n_1 + \cdots + n_s + s$ variables $z_{i,j}$, $1 \leq i \leq s$, $0 \leq j \leq n_s$. A polynomial $f \in R$ is said to be multihomogeneous with multidegree $(a_1, \dots, a_s)$ (each a_i being a non-negative integer) if for all i it is homogeneous of degree a_i as a polynomial in the variables $z_{i,0}, \dots, z_{i,n_i}$. $\mathbf{P}^{n_1}(q) \times \cdots \times \mathbf{P}^{n_s}_(q)$ is the natural setting for the study of multihomogeneous polynomals of $GF(q)[z_{i,j}]$ and the methods used in the other sections for homogeneous polynomials may be used to solve interpolation problems for multihomogeneous polynomials over $GF(q)$. Let $W(n_1, d_1; \dots; n_s, d_s)$ denotes the vector space of all multihomogeneous polynomials over $GF(q)$ with multidegree $(a_1, \dots, a_s)$. Notice that $\dim(W(n_1, d_1; \dots; n_s, d_s)) = \prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$. Set $x_{i,j} := z_{i,j}/z_{i,0}$, $1 \leq i \leq s$, $1 \leq j \leq n_i$. We will fix the coordinates $x_{i,j}$, $1 \leq i \leq s$, $1 \leq j \leq n_i$ on $\mathbb{A}^{n_1+\cdots+n_s}$ and write $\partial_{i,j}$ instead of $\partial/\partial x_{i,j}$. If we see $\mathbb{A}^{n_1+\cdots+n_s}(q)$ as a subset of $PG(n_1, q) \times \cdots \times PG(n_s, q)$, all the interpolations problem we considered in this paper may be see as interpolation problems for multihomogeous polynomials. Let $\pi_i : \mathbf{P}_K^{n_1} \times \cdots \times \mathbf{P}_K^{n_s} \to \mathbf{P}_K^{n_i}$ be the projection. Set $\mathcal{O}_{\mathbf{P}_K^{n_1} \times \cdots \times \mathbf{P}_K^{n_s}}(a_1, \dots, a_s) := \bigotimes_{i=1}^{s} \pi_i^*(\mathcal{O}_{\mathbf{P}_K^{n_i}}(a_i))$. Hence $\mathcal{O}_{\mathbf{P}_K^{n_1} \times \cdots \times \mathbf{P}_K^{n_s}}(a_1, \dots, a_s)$ is a line bundle on $\mathbf{P}_K^{n_1+\cdots+n_s}$. Using Künneth formula and the cohomology of the projective spaces we immediately get the following remarks. If $a_i \ge 0$ for all i, then $H^0(\mathbf{P}_K^{n_1+\cdots+n_s}, \mathcal{O}_{\mathbf{P}_K^{n_1}\times\cdots\times\mathbf{P}_K^{n_s}}(a_1,\dots,a_s))$ is the K-vector spaces of all multihomogeneous polynomials with multidegree $(a_1,\cdots,a_s)$. If $a_i < 0$ for some i, then $H^0(\mathbf{P}_K^{n_1+\cdots+n_s}, \mathcal{O}_{\mathbf{P}_K^{n_1}\times\cdots\times\mathbf{P}_K^{n_s}}(a_1,\dots,a_s)) = \{0\}$. If $a_i \ge -1$ for all i, then $H^1(\mathbf{P}_K^{n_1+\cdots+n_s}, \mathcal{O}_{\mathbf{P}_K^{n_1}\times\cdots\times\mathbf{P}_K^{n_s}}(a_1,\dots,a_s)) = \{0\}$.

As an example we will give the following extension of Theorem 1. To avoid a cumbersome notation we avoid here the subscript $GF(q)$ and write $\mathcal{O}_{(n_1,\dots,n_s)}(a_1,\dots,a_n)$ instead of $\mathcal{O}_{\mathbf{P}^{n_1}\times\cdots\mathbf{P}^{n_s}}(a_1,\dots,a_s)$.

Theorem 3. *Fix a prime power q and integers $s \ge 1$, $n_i > 0$, $0 < d_i < q$,, $1 \le i \le s$, $b \ge 0$ and $a \ge 0$. Then the are $A, B \subset \mathbb{A}(q)^{n_1+\cdots+n_s}$ and for each $P \in B$ a pair $(i(P), j(P))$ with $1 \le i(P) \le s$, $1 \le j(P) \le n_{i(P)}$ such that* $\mathrm{card}(A) = a$, $\mathrm{card}(B) = b$, *$A \cap B = \emptyset$ and $h^0(\mathbf{P}^{n_1+\cdots+n_s}, \mathcal{I}_Z \otimes O_{(n_1,\dots,n_s)}(d_1,\dots,d_n)) = \max\{0, \prod_{i=1}^{s}\binom{n_i+d_i}{n_i} - a - 2b\}$, where Z is the zero-dimensional scheme (or the interpolation problem) associated to the points of $A \cup B$ and to all partial derivatives $\partial_{i(P),j(P)}$ for all $P \in B$, i.e. the interpolation problem associated to Z gives the maximal possible number of independent conditions to the vector space of all multihomogeneous polynomials with degree d_i for the homogeneous coordinates $z_{i,j}$, $0 \le j \le n_i$.*

2. The Proofs

Remark 2. For all integer d such that $0 \le d < q$ we have $V_{n,d}(-\mathbb{A}^n(q)) = \{0\}$ and $W_{n,d}(-PG(n,q)) = \{0\}$. Notice that this is false for $d \ge q$ (e.g use $x_1^{d-q}(x_1^q - x_1)$). Hence there are $S \subset \mathbb{A}^n(q)$ and $M \subset PG(n,q)$ such that $\mathrm{card}(S) = \mathrm{card}(M) = \binom{n+d}{n}$ and $V_{n,d}(-S) = W_{n,d}(-M) = \{0\}$. Hence S (resp. M) imposes independent conditions to $V_{n,d}$ (resp. $W_{n,d}$). Hence for all $A \subseteq S$, all $B \subseteq M$, all $S \subseteq E \subseteq \mathbb{A}^n(q)$ and all $M \subseteq F \subseteq PG(n,q)$ we have $\dim(V_{n,d}(-A)) = \binom{n+d}{n} - \mathrm{card}(A)$, $\dim(W_{n,d}(-B)) = \binom{n+d}{n} - \mathrm{card}(B)$, $V_{n,d}(-E) = \{0\}$ and $W_{n,d}(-F) = \{0\}$.

For any algebraic scheme X over a field and any closed subscheme Y of X let $\mathcal{I}_{Y,X}$ (or just $\mathcal{I}_Y$ if X is either a projective space or an affine space and there is no danger of misunderstandings) denote the ideal sheaf of Y in X. Let $D \subset X$ be an effective Cartier divisor of X. We recall that the residual scheme $\mathrm{Res}_D(Y)$ of Y is the closed subscheme of X with ideal sheaf $\mathcal{I}_{\mathrm{Res}_D(Y),X} := (\mathcal{I}_{D,X} : \mathcal{I}_{Y,X})$. For instance, for any $P \in D_{reg}$ and any integer $m > 0$ we have $\mathrm{Res}_D(mP) = (m-1)P$ (with the convention $0P := \emptyset$).

Remark 3. For any $L \in \mathrm{Pic}(X)$ we have the following exact sequence of sheaves on X:

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Y)},X} \otimes L \to \mathcal{I}_{Y,X} \otimes L \to \mathcal{I}_{Y\cap D,D} \otimes (L_{|D}) \to 0\,. \qquad (1)$$

From the long cohomology exact sequence of the exact sequence (1) we immediately get the following two assertions which are an elementary form of the so-called Horace Lemma introduced by A. Hirschowitz:

(a) $h^0(X, \mathcal{I}_{Y,X} \otimes L) \le h^0(X, \mathcal{I}_{\mathrm{Res}_{D(Y)},X} \otimes L) + h^0(D, \mathcal{I}_{Y\cap D,D} \otimes (L_{|D}))$;

(b) $h^0(X, \mathcal{I}_{Y,X} \otimes L) \le h^0(X, \mathcal{I}_{\mathrm{Res}_{D(Y)},X} \otimes L) + h^0(D, \mathcal{I}_{Y\cap D,D} \otimes (L_{|D}))$.

Proof of Theorem 1. The case $a = 0$ is true by Remark 2. Since the case $n = 1$ is obvious, we may assume $n > 1$ and assume that the statement is true in $\mathbb{A}(q)^{n-1}$ for all non-negative integers a', b', d' with $a' + b' \le q^{n-1}$ and $d' < q$. Taking $i(P) \ne n$ the case $n' := n - 1$, $d' := d$, $a' = a$ and $b' := b$ covers the statement for the integers n, d, a, b if $a + 2b \le \binom{n+d-1}{n-1}$. Now assume $\binom{n+d-1}{n-1} \le a + 2b \le \binom{n+d}{n}$. We use induction on the integer d, the case $d = 1$ being obvious. Assume $d \ge 2$. Let $H \subset \mathbb{A}^n(q)$ be the hyperplane $\{x_n = 0\}$. We apply the inductive assumption on n to H. Set $b_1 := \min\{b, [(\binom{n+d-1}{n-1})/2\}$ and $a_1 := \binom{n+d-1}{n-1} - 2b_1$. Since $a + 2b \ge \binom{n+d-2}{n-1}$, we have $a_1 \ge 0$ and we may apply the inductive assumption on n to H with respect to the data $n-1, d, a_1, b_1$. Choose $A_1, B_1 \in H$ giving a solution for these data and some indices $i(Q) \in \{1, \ldots, n-1\}$, $Q \in B_1$. If either $b_1 = b$ or $\binom{n+d-1}{n-1}$ is even, call W the interpolation data on $\mathbb{A}^n(q)$ given by A_1, B_1 and the derivatives $\delta_{i(Q)}$ at each $Q \in B_1$. In this case we have $a_1 + 2b_1 = \binom{n+d-1}{n-1}$ and the data W satisfies $V_{n-1,d}(-W)) = \{0\}$. Furthermore, the empty data is the residual interpolation data of W with respect to the hyperplane H. Since $a - a_1 + 2(b - b_1) \le \binom{n+d-1}{n}$, we conclude by induction on the integer d and Horace Lemma (i.e. Remark 3). Now assume $b_1 < b$ and $\binom{n+d-1}{n-1}$ odd. Notice that $a_1 > 0$ in this case. Fix any $P \in A_1$ and let W' be the interpolation data for n, d obtaining from W assigning also the partial derivatives ∂_n at the point P. Hence W' has parameters $(a_1 - 1, b_1 + 1)$. The residual data of W' with respect to H is just the point P with no derivatives assigned. Since $a - a_1 + 1 + 2(b - b_1) \le \binom{n+d-1}{n}$ in this case, we conclude by induction on the integer d using again Horace Lemma, i.e. Remark 3. Now assume $a + 2b > \binom{n+d}{n}$. Set $b'' := \min\{b, [\binom{n+d}{n}/2]\}$ and $a'' := \binom{n+d}{n} - 2b''$. Hence $b'' \le b$ and $a'' + b'' \le a + b$. The truth of case with data n, d, a'', b'' gives the truth of the case with data n, d, a, b. $\square$

Proof of Theorem 3. Fix q. Since the case $s = 1$ is covered by Theorem 2, we may assume $s \geq 2$ and that the statement is true for the integer $s' := s - 1$ and all data n'_i, d'_i, $1 \leq i \leq s-1$ and a', b'. We fix $P \in PG(n_s, q)$ and we see $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{s-1}}$ as the slice $H := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{s-1}} \times \{P\}$ of $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$. We use induction on the integer n_s, starting from the case $n_s = 0$, i.e. $s' = s-1$. For each value of the integer n_s we use induction on the integer d_s. With these small modifications, the proof of Theorem 2 may be very easily adapted. Just to help to the reader (and to see why with this appoach we cannot prescribe in advance which pair $(i(P), j(P))$ we need at each $P \in B$, i.e. to point out the main weakness of this proof), we write down the first inductive step: $n_s = 1$ and $d_s = 1$. Just to fix the notation we also assume $a + 2b \leq 2\prod_{i=1}^{s-1} \binom{n_i+d_i}{n_i}$. Set $b' := \min\{0, [\prod_{i=1}^{s-1} \binom{n_i+d_i}{n_i}/2]\}$ and $a' := \prod_{i=1}^{s-1} \binom{n_i+d_i}{n_i} - 2b'$. Hence either $0 \leq a' \leq 1$ or $b' = b$. We choose a solution, Z', for H and the data d_j, $1 \leq j \leq s-1$, a', b'. Hence for these data we always have $i(P) \leq s-1$. If $b' < b$ and $a' = 1$ we prescribe $i(P) = s$ at the unique $P \in Z'_{red}$ supporting the reduced connected component of Z'. Then we take another parallel H' and we put here the remaing data, always prescribing $i(P) = s$, but (following the proof of Theorem 2) not knowing in advance the corresponding value of $j(P)$. Then we use Horace Lemma (Remark 3) for this data with respect to H. Notice that the corresponding part of the residual scheme contained in H is either $\emptyset$, or, in the case $b' < b$ and $a' = 1$, just one point and we may choose arbitrarly this point in the set of all $GF(q)$-points of H. $\square$

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References

[1] E. Ballico, Interpolation of homogeneous polynomials over a finite field, *Preprint.*

[2] A. Hefez, Nonreflexive curves, *Compositio Math.*, **69**, No. 1 (1989), 3-35.