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ON GENERAL PRINCIPLES OF ISHIKAWA ITERATIVE SCHEME WITH ERRORS OF MULTI-VALUED MAPPINGS IN NORMED LINEAR SPACES

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Abstract: In this paper we establish a few generic theorems of the Ishikawa iterative scheme with errors for a pair of multi-valued mappings in normed linear spaces. The results presented in this paper generalize the corresponding results due to Beg-Azam, Guay-Singh, Hu-Huang-Rhoades, Liu-Kang-Ume and Rashwan-Saddeek.

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1. Introduction and Preliminaries

Guay-Singh [2] and Rashwan-Saddeek [5] established convergence of the Ishikawa iterative schemes for a pair of single-valued contractive type mappings. Beg-Azam [1] extended the results due to Guay-Singh [2] and Rashwan-Saddeek

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[5] to multi-valued mappings. Hu-Huang-Rhoades [3] studied convergence of the Ishikawa iterative scheme for a pair of multi-valued mappings. Recently, Liu-Kang-Ume [4] investigated convergence of the Ishikawa iterative scheme for a pair of multi-valued mappings, which extend the results of Guay-Singh [2], Hu-Huang-Rhoades [3] and Rashwan-Saddeek [6].

The purpose of this paper is to establish a few generic theorems of the Ishikawa iterative scheme with errors for a pair of multi-valued mappings in normed linear spaces. The results presented in this paper generalize the corresponding results due to Beg-Azam [1], Hu-Huang-Rhoades [3], Liu-Kang-Ume [4] and Rashwan-Saddeek [6].

In what follows, we assume that (X, d) is a metric space, $CB(X)$ the collection of nonempty, closed and bounded subsets of X, and $H(A, B)$ is the Hausdorff metric on X. Define $d(x, A) = \inf\{d(x, a) : a \in A\}$ for any $A \subseteq X$ and $x \in X$.

Now we recall the following result and concept.

Lemma 1.1. (see Nadler [5]) *Suppose that A, B are in $CB(X)$ and a is in A. Then, for any $\epsilon > 0$, there exists some $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Definition 1.1. Let K be a nonempty convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings. For any $x_0 \in K$, the sequence $\{x_n\}_{n\geq 0}$ defined by

$$\begin{aligned} y_n &= a'_n x_n + b'_n t_n + c'_n v_n, \quad t_n \in Tx_n,\ n \geq 0, \\ x_{n+1} &= a_n x_n + b_n s_n + c_n u_n, \quad s_n \in Sy_n,\ n \geq 0, \end{aligned} \tag{1.1}$$

is called the *Ishikawa iterative scheme with errors*, where

$$\{u_n\}_{n\geq 0} \text{ and } \{v_n\}_{n\geq 0} \text{ are arbitrary bounded sequences in } K, \tag{1.2}$$

$\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$, $\{a'_n\}_{n\geq 0}$, $\{b'_n\}_{n\geq 0}$ and $\{c'_n\}_{n\geq 0}$ are real sequences with

$$0 \leq a_n,\ b_n,\ c_n, \quad a'_n,\ b'_n,\ c'_n \leq 1, \quad n \geq 0, \tag{1.3}$$

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad n \geq 0. \tag{1.4}$$

It is clear that the Ishikawa iterative schemes can be obtained from the Ishikawa iterative scheme with errors by setting $c_n = c'_n = 0$ for $n \geq 0$. It follows from Lemma 1.1 that there exist $t_n \in Tx_n$ and $s_n \in Sy_n$ such that

$$\|t_n - s_n\| \leq H(Tx_n, Sy_n) + \epsilon_n, \quad n \geq 0, \tag{1.5}$$

where

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \text{and} \quad \epsilon_n > 0, \quad n \geq 0. \tag{1.6}$$

Let R^+ denote the set of nonnegative real numbers and

$$\begin{aligned} \Phi &= \{\phi : \phi : (R^+)^2 \to R^+ \text{ satisfies (1.7), (1.8) and (1.9)}\}, \\ \Psi &= \{\psi : \psi : R^+ \to R^+ \text{ satisfies that } \psi(t) < t \text{ for any } t > 0\}, \end{aligned}$$

where

$$\phi(x, y) \text{ is nondecreasing in each coordinate variable,} \tag{1.7}$$

$$\phi(0, t) < t, \text{ for each } t > 0, \text{ and} \tag{1.8}$$

$$\limsup_{n\to\infty} \phi(x_n, y_n) \leq \phi\Big(\limsup_{n\to\infty} x_n, \limsup_{n\to\infty} y_n\Big), \tag{1.9}$$

for any sequences $\{x_n\}_n \geq 0$ and $\{y_n\}_{n\geq 0}$ in R^+.

2. Main Results

Theorem 2.1. *Let K be a nonempty closed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings. Suppose that the Ishikawa iterative scheme with errors* (1.1) *with $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$, $\{a'_n\}_{n\geq 0}$, $\{b'_n\}_{n\geq 0}$ and $\{c'_n\}_{n\geq 0}$ satisfying* (1.3), (1.4)

$$\liminf_{n\to\infty} b_n = b > 0, \tag{2.1}$$

$$\lim_{n\to\infty} c_n = 0, \tag{2.2}$$

$\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ satisfying (1.2), *and $\{s_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$ and $\{\epsilon_n\}_{n\geq 0}$ satisfying* (1.5) *and* (1.6) *converges strongly to a point p. If there exists $\phi \in \Phi$ and a nonnegative sequence $\{r_n\}_{n\geq 0}$ with*

$$\lim_{n\to\infty} r_n = 0, \tag{2.3}$$

such that, for all n sufficiently large, S and T satisfy

$$H(Tx_n, Sy_n) \leq \phi(\|x_n - s_n\|, \|x_n - t_n\|) + r_n \tag{2.4}$$

and

$$\begin{aligned} H(Sp, Tx_n) \leq\ & \phi(\max\{\|x_n - p\|, d(p, Tx_n), d(x_n, Tx_n)\}, \\ & \max\{d(p, Sp), d(x_n, Sp)\}), \end{aligned} \tag{2.5}$$

then p is a fixed point of S. Furthermore, if there exists an $\psi \in \Psi$ such that

$$H(Tp, Sp) \le \psi(d(p, Tp) + d(p, Sp)), \tag{2.6}$$

then p is a common fixed point of S and T.

Proof. Using (1.1) and (1.4), we know that, for all $n \ge 0$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n(x_n - p) + b_n(s_n - p) + c_n(u_n - p)\| \\ &\ge b_n\|s_n - p\| - a_n\|x_n - p\| - c_n\|v_n - p\|, \end{aligned}$$

which implies that

$$b_n\|s_n - p\| \le \|x_{n+1} - p\| + a_n\|x_n - p\| + c_n\|u_n - p\|\,, \tag{2.7}$$

for any $n \ge 0$. Since $\lim_{n\to\infty} x_n = p$, it follows from (1.2), (1.3), (2.1), (2.2) and (2.7) that

$$\lim_{n\to\infty} \|s_n - p\| = 0. \tag{2.8}$$

In the light of (1.5), (1.7) and (2.4), we arrive at

$$\begin{aligned} \|t_n - p\| &\le \|t_n - s_n\| + \|s_n - p\| \\ &\le H(Tx_n, Sy_n) + \epsilon_n + \|s_n - p\| \\ &\le \phi(\|x_n - s_n\|, \|x_n - t_n\|) + r_n + \epsilon_n + \|s_n - p\| \\ &\le \phi(\|x_n - p\| + \|s_n - p\|, \|x_n - p\| + \|t_n - p\|) \\ &\quad + r_n + \epsilon_n + \|s_n - p\|\,, \end{aligned} \tag{2.9}$$

for all n sufficiently large. By virtue of $\lim_{n\to\infty} x_n = p$, (1.6), (1.7), (1.9), (2.3), (2.8) and (2.9), we easily conclude that

$$\begin{aligned} &\limsup_{n\to\infty} \|t_n - p\| \\ \le\ &\limsup_{n\to\infty} \phi(\|x_n - p\| + \|s_n - p\|, \|x_n - p\| + \|t_n - p\|) \\ &+ \limsup_{n\to\infty}(r_n + \epsilon_n + \|s_n - p\|) \\ \le\ &\phi\Big(\limsup_{n\to\infty} \phi(\|x_n - p\| + \|s_n - p\|), \limsup_{n\to\infty}(\|x_n - p\| + \|t_n - p\|)\Big) \\ \le\ &\phi\Big(0, \limsup_{n\to\infty} \|t_n - p\|\Big). \end{aligned} \tag{2.10}$$

It follows from (1.8) and (2.10) that $\limsup_{n\to\infty} \|t_n - p\| = 0$, which means that

$$\lim_{n\to\infty} t_n = p. \tag{2.11}$$

Using (2.5) and $t_n \in Tx_n$, $s_n \in Sy_n$, we deduce that

$$\begin{aligned} d(p,Sp) &\leq \|t_n - p\| + d(t_n, Sp) \\ &\leq \|t_n - p\| + H(Tx_n, Sp) \\ &\leq \|t_n - p\| + \phi(\max\{\|x_n - p\|, d(p, Tx_n), d(x_n, Tx_n)\}, \\ &\quad \max\{d(p,Sp), d(x_n, Sp)\}) \\ &\leq \|t_n - p\| + \phi(\max\{\|x_n - p\|, \|t_n - p\|, \|x_n - t_n\|\}, \\ &\quad \|x_n - p\| + d(p, Sp)), \end{aligned} \tag{2.12}$$

for all n sufficiently large. According to $\lim_{n\to\infty} x_n = p$, (1.7), (1.9), (2.8), (2.11) and (2.12), we derive that

$$\begin{aligned} &d(p,Sp) \\ \leq\ &\limsup_{n\to\infty} \phi(\max\{\|x_n - p\|, \|t_n - p\|, \|x_n - t_n\|\}, \|x_n - p\| + d(p,Sp)) \\ \leq\ &\phi(0, d(p,Sp)). \end{aligned} \tag{2.13}$$

It follows from (1.8) and (2.13) that $p \in Sp$. In view of (2.6), we have

$$d(p,Tp) \leq H(Tp,Sp) \leq \psi(d(p,Tp) + d(p,Sp)) = \psi(d(p,Tp)),$$

which implies that $p \in Tp$. This completes the proof. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings. Suppose that the Ishikawa iterative scheme with errors* (1.1) *with $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$, $\{a'_n\}_{n\geq 0}$, $\{b'_n\}_{n\geq 0}$ and $\{c'_n\}_{n\geq 0}$ satisfying* (1.3), (1.4), (2.1), (2.2) *and*

$$\lim_{n\to\infty} c'_n = 0, \tag{2.14}$$

$\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ satisfying (1.2), *and $\{s_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$ and $\{\epsilon_n\}_{n\geq 0}$ satisfying* (1.5) *and* (1.6) *converges strongly to a point p. If there exists an $\phi \in \Phi$ and a nonnegative sequence $\{r_n\}_{n\geq 0}$ such that, for all n sufficiently large, S and T satisfy*

$$H(Tx_n, Sy_n) \leq \phi(\|x_n - s_n\|, \max\{\|x_n - t_n\|, \|y_n - t_n\|\}) + r_n, \tag{2.15}$$

(2.3) *and* (2.5), *then p is a fixed point of S. Furthermore, if there exists an $\psi \in \Psi$ such that* (2.6) *holds, then p is a common fixed point of S and T.*

Proof. As in the proof of Theorem 2.1, we conclude that (2.8) holds. It follows from (1.5), (1.7) and (2.15) that

$$\begin{aligned}
& \|t_n - p\| \\
\le\ & \|t_n - s_n\| + \|s_n - p\| \\
\le\ & H(Tx_n, Sy_n) + \epsilon_n + \|s_n - p\| \\
\le\ & \phi(\|x_n - s_n\|, \max\{\|x_n - t_n\|, \|y_n - t_n\|\}) + r_n + \epsilon_n + \|s_n - p\| \\
\le\ & \phi(\|x_n - s_n\|, \max\{\|x_n - p\| + \|t_n - p\|, a'_n\|x_n - t_n\| \\
& + c'_n\|v_n - t_n\|\}) + r_n + \epsilon_n + \|s_n - p\| \\
\le\ & \phi(\|x_n - s_n\|, \max\{\|x_n - p\| + \|t_n - p\|, a'_n\|x_n - p\| \\
& + (a'_n + c'_n)\|t_n - p\| + c'_n\|v_n - p\|\}) + r_n + \epsilon_n + \|s_n - p\| \\
\le\ & \phi(\|x_n - s_n\|, \max\{\|x_n - p\| + \|t_n - p\|, a'_n\|x_n - p\| + \|t_n - p\| \\
& + c'_n\|v_n - p\|\}) + r_n + \epsilon_n + \|s_n - p\|\,, \qquad (2.16)
\end{aligned}$$

for all n sufficiently large. Since $\lim_{n\to\infty} x_n = p$, by (1.2), (1.6), (1.7), (1,9), (2.3), (2.8), (2.14) and (2.16), we see that

$$\limsup_{n\to\infty} \|t_n - p\| \le \phi\Big(0, \limsup_{n\to\infty} \|t_n - p\|\Big),$$

which gives that $\lim_{n\to\infty} \|t_n - p\| = 0$. The rest of the proof is exactly the same as that of Theorem 2.1. This completes the proof. □

Remark 2.1. In case $c_n = c'_n = 0$ for $n \ge 0$, then Theorem 2.1 and Theorem 2.2 reduce to two results due to Liu-Kang-Ume [4], in turn, which extend Theorem 1.1 of Guay-Singh [2], Theorem 1 and Theorem 2 of Hu-Huang-Rhoades [3] and Theorem 2.2 of Rashwan-Saddeek [6].

Theorem 2.3. *Let K be a nonempty closed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings satisfying*

$$\begin{aligned}
& H(Tx, Sy) \\
& \qquad \le q \max\{k\|x - y\|, d(x, Tx) + d(y, Sy), d(x, Sy) + d(y, Tx)\}\,, \quad (2.17)
\end{aligned}$$

for all $x, y \in K$, where $k \ge 0$ and $0 < q < 1$. If there exists a point $x_0 \in K$ such that $\{x_n\}_{n\ge0}$ satisfying (1.1)-(1.6), (2.1), (2.2), (2.14) *and*

$$\lim_{n\to\infty} b'_n = 0 \qquad (2.18)$$

converges strongly to a point p, then p is a common fixed point of S and T.

Proof. According to (1.1), (1.4) and (2.17), we deduce that

$$\begin{aligned}
&H(Tx_n, Sy_n)\\
&\quad \le q\max\{k\|x_n - y_n\|, d(x_n, Tx_n) + d(y_n, Sy_n), d(x_n, Sy_n) + d(y_n, Tx_n)\}\\
&\qquad \le q\max\{kb'_n\|x_n - t_n\| + kc'_n\|v_n - x_n\|, \|x_n - t_n\| + \|y_n - s_n\|,\\
&\qquad\qquad \|x_n - s_n\| + \|y_n - t_n\|\}\\
&\qquad\quad \le q\max\{kb'_n\|x_n - t_n\| + kc'_n\|v_n - x_n\|, (1 + b'_n)\|x_n - t_n\|\\
&\qquad\qquad + c'_n\|v_n - x_n\| + \|x_n - s_n\|, \|x_n - s_n\| + \|y_n - t_n\|\}\\
&\quad\ \le q\|x_n - s_n\| + \max\{qkb'_n, q(1 + b'_n)\}\max\{\|x_n - t_n\|, \|y_n - t_n\|\}\\
&\qquad\qquad + \max\{1, k\}qc'_n\|v_n - x_n\|, \qquad (2.19)
\end{aligned}$$

$$\begin{aligned}
&H(Tx_n, Sp)\\
\le\ & q\max\{k\|x_n - p\|, d(x_n, Tx_n) + d(p, Sp), d(x_n, Sp) + d(p, Tx_n)\}\\
\le\ & q\max\{k\|x_n - p\|, d(x_n, Tx_n), d(p, Tx_n)\}\\
& + q\max\{d(p, Sp), d(x_n, Sp)\}, \qquad (2.20)
\end{aligned}$$

for all $n \ge 0$ and

$$\begin{aligned}
H(Tp, Sp) &\le q\max\{0, d(p, Tp) + d(p, Sp), d(p, Sp) + d(p, Tp)\}\\
&= q[d(p, Tp) + d(p, Sp)]. \qquad (2.21)
\end{aligned}$$

Let $r_n = \max\{1, k\}qc'_n\|v_n - x_n\|$ for each $n \ge 0$. Since $\{x_n\}_{n\ge 0}$ converges strongly to p and $\{v_n\}_{n\ge 0}$ is bounded, by (2.14), we get that $\lim_{n\to\infty} r_n = 0$. In view of (2.18), there exists $\beta \in (q, 1)$ satisfying

$$\max\{qkb'_n, q(1 + b'_n)\} < \beta, \qquad (2.22)$$

for all n sufficiently large. Set $\phi(x, y) = qx + \beta y$ and $\psi(x) = qx$. It follows from (2.19)-(2.22) that (2.3), (2.5), (2.6) and (2.15) are satisfied. By Theorem 2.2, we know that p is a common fixed point of S and T. This completes the proof. □

Remark 2.2. Theorem 1 of Beg-Azam [1] and Corollary 1 of Hu-Huang-Rhoades [3] are special cases of Theorem 2.3.

Theorem 2.4. *Let K be a nonempty colsed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings* *satisfying*

$$H(Tx,Sy) \leq \max\left\{\|x-y\|, \frac{1}{2}[d(x,Tx)+d(y,Sy)], \frac{1}{2}[d(x,Sy)+d(y,Tx)]\right\}, \quad (2.23)$$

for all $x, y \in K$. If there exists a point $x_0 \in K$ such that $\{x_n\}_{n\geq 0}$ satisfying (1.1)-(1.6), (2.1), (2.2), (2.14) *and*

$$\limsup_{n\to\infty} b'_n < 1 \quad (2.24)$$

converges strongly to a point p, then p is a common fixed point of S and T.

Proof. It is sufficient to show that (2.3), (2.5), (2.6) and (2.15) are fulfilled. It follows from (1.1), (1.4) and (2.23) that

$$\begin{aligned} &H(Tx_n, Sy_n) \\ &\leq \max\left\{\|x_n-y_n\|, \frac{1}{2}[d(x_n,Tx_n)+d(y_n,Sy_n)], \frac{1}{2}[d(x_n,Sy_n)+d(y_n,Tx_n)]\right\} \\ &\qquad \leq \max\left\{b'_n\|x_n-t_n\|+c'_n\|v_n-x_n\|, \frac{1}{2}[(1+b'_n)\|x_n-t_n\| \right. \\ &\qquad \left. + c'_n\|v_n-x_n\|+\|x_n-s_n\|], \frac{1}{2}[\|x_n-s_n\|+\|y_n-t_n\|]\right\} \\ &\quad \leq \frac{1}{2}\|x_n-s_n\|+\frac{1}{2}(1+b'_n)\max\{\|x_n-t_n\|,\|y_n-t_n\|\}+c'_n\|v_n-x_n\|, \end{aligned}$$

$$\begin{aligned} &H(Tx_n, Sp) \\ &\quad \leq \max\left\{\|x_n-p\|, \frac{1}{2}[d(x_n,Tx_n)+d(p,Sp)], \frac{1}{2}[d(x_n,Sp)+d(p,Tx_n)]\right\} \\ &\quad \leq \max\left\{\|x_n-p\|, d(x_n,Tx_n), d(p,Tx_n)\} + \frac{1}{2}\max\{d(p,Sp), d(x_n,Sp)\right\}, \end{aligned}$$

for all $n \geq 0$, and

$$\begin{aligned} H(Tp,Sp) &\leq \max\left\{0, \frac{1}{2}[d(p,Tp)+d(p,Sp)], \frac{1}{2}[d(p,Sp)+d(p,Tp)]\right\} \\ &= \frac{1}{2}[d(p,Tp)+d(p,Sp)]. \end{aligned}$$

Set $r_n = c'_n\|v_n - x_n\|$ for all $n \geq 0$. It is easy to verify that $\lim_{n\to\infty} r_n = 0$. (2.24) yields that there exists some $\beta \in \left(\limsup_{n\to\infty} b'_n, 1\right)$ such that $\frac{1}{2}(1+b'_n) < \beta$ for all n sufficiently large. Put $\phi(x,y) = \frac{1}{2}x + \beta y$ and $\psi(x) = \frac{1}{2}x$. Thus (2.3), (2.5), (2.6) and (2.15) are fulfilled. This completes the proof. ☐

Theorem 2.5. *Let K be a nonempty colsed convex subset of a normed linear space $(X, \|\cdot\|)$ and let $S, T : K \to CB(K)$ be multi-valued mappings satisfying*

$$H(Tx, Sy) \leq q \max\left\{\|x-y\|, \frac{d(y,Sy)[1+d(x,Tx)]}{1+\|x-y\|}, \frac{d(x,Sy)[1+d(x,Tx)+d(y,Tx)]}{2[1+\|x-y\|]}\right\}, \quad (2.25)$$

for all $x, y \in K$, where $0 < q < 1$. If there exists a point $x_0 \in K$ such that $\{x_n\}_{n\geq 0}$ satisfying (1.1)-(1.6), (2.1), (2.2), (2.14) *and* (2.18) *converges strongly to a point p, then p is a common fixed point of S and T.*

Proof. As in the proof of Theorem 2.1, we obtain that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = p. \quad (2.26)$$

Put $r_n = qc'_n(\|v_n - x_n\| + \|v_n - t_n\|)$. Thus (1.2), (2.14), (2.18), (2.26) and $\lim_{n\to\infty} x_n = p$ ensure that there exists some β such that

$$\|x_n - s_n\| < 1, \quad (2.27)$$

$$\max\left\{1+b'_n, \frac{1+\|x_n-t_n\|}{1+\|x_n-p\|}, \frac{1+\|x_n-t_n\|+\|t_n-p\|}{2(1+\|x_n-p\|)}\right\} < \beta < 1, \quad (2.28)$$

for all n sufficiently large and

$$\lim_{n\to\infty} r_n = 0. \quad (2.29)$$

Observe that (1.1), (1.4), (2.25), (2.27) and (2.28) yield that

$$\begin{aligned}
& H(Tx_n, Sy_n) \\
\le\ & q \max\left\{ \|x_n - y_n\|, \frac{d(y_n, Sy_n)[1 + d(x_n, Tx_n)]}{1 + \|x_n - y_n\|}, \right. \\
& \left. \frac{d(x_n, Sy_n)[1 + d(x_n, Tx_n) + d(y_n, Tx_n)]}{2(1 + \|x_n - y_n\|)} \right\} \\
\le\ & q \max\left\{ b'_n\|x_n - t_n\| + c'_n\|v_n - x_n\|, \right. \\
& \frac{(b'_n\|x_n - t_n\| + c'_n\|v_n - x_n\| + \|x_n - s_n\|)(1 + \|x_n - t_n\|)}{1 + b'_n\|x_n - t_n\| + c'_n\|v_n - x_n\|}, \\
& \left. \frac{\|x_n - s_n\|(1 + \|x_n - t_n\| + a'_n\|x_n - t_n\| + c'_n\|v_n - t_n\|)}{2(1 + b'_n\|x_n - t_n\| + c'_n\|v_n - x_n\|)} \right\} \\
\le\ & q \max\{ b'_n\|x_n - t_n\| + c'_n\|v_n - x_n\|, \\
& (1 + b'_n)\|x_n - t_n\| + c'_n\|v_n - x_n\| + \|x_n - s_n\|, \\
& \frac{1}{2}[\|x_n - s_n\| + (1 + a'_n)\|x_n - t_n\| + c'_n\|v_n - t_n\|]\} \\
\le\ & q\|x_n - s_n\| + q(1 + b'_n)\|x_n - t_n\| + qc'_n(\|v_n - x_n\| + \|v_n - t_n\|) \\
\le\ & q\|x_n - s_n\| + \|x_n - t_n\| + r_n, \qquad (2.30)
\end{aligned}$$

and

$$\begin{aligned}
& H(Tx_n, Sp) \\
\le\ & q \max\left\{ \|x_n - p\|, \frac{d(p, Sp)(1 + \|x_n - t_n\|)}{1 + \|x_n - p\|}, \right. \\
& \left. \frac{d(x_n, Sp)[1 + \|x_n - t_n\| + \|p - t_n\|]}{2(1 + \|x_n - p\|)} \right\} \\
\le\ & q\|x_n - p\| + q \max\left\{ \frac{1 + \|x_n - t_n\|}{1 + \|x_n - p\|}, \frac{1 + \|x_n - t_n\| + \|t_n - p\|}{2(1 + \|x_n - p\|)} \right\} \\
& \times \max\{d(p, Sp), d(x_n, Sp)\} \\
\le\ & q\|x_n - p\| + \beta \max\{d(p, Sp), d(x_n, Sp)\}, \qquad (2.31)
\end{aligned}$$

for all n sufficiently large. Set $\phi(x, y) = qx + \beta y$. It follows from (2.30) and (2.31) that (2.5) and (2.15) are satisfied. It follows from Theorem 2.2 that p is a fixed point of S. By virtue of (2.25), we have

$$\begin{aligned} & H(Sp,Tp) \\ \leq\ & q\max\Big\{0, d(p,Sp)[1+d(p,Tp)], \frac{1}{2}d(p,Sp)[1+d(p,Tp)+d(p,Sp)]\Big\} \\ =\ & 0. \end{aligned}$$

That is, (2.6) is fulfilled trivially. Thus p is a common fixed points of S and T by Theorem 2.2. This completes the proof. □

Remark 2.3. Theorem 2.5 generalizes Corollary 3 of Hu-Huang-Rhoades [3].

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