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ITERATIVE ALGORITHM FOR GENERALIZED NONLINEAR MIXED QUASIVARIATIONAL INEQUALITIES

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Abstract: In this paper, we use the auxiliary principle technique to suggest a class of predictor-corrector methods for solving generalized nonlinear mixed quasivariational inequalities. The convergence of the proposed methods only requires the continuity and the g-partially relaxed strong monotonicity of mappings. The results presented in this paper extend, improve and unify the corresponding results in this area.

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1. Introduction

One of the most interesting and important problems in the variational inequality theory is the development of an efficient iterative algorithm to compute approximate solutions and the convergence analysis of the algorithm, see [1]-[12]. One of the most effective numerical technique is the projection method and its variant forms. It is well known that the convergence analysis of the projection method requires that the underlying operator must strongly monotone and Lipschitz continuous. These strict conditions rule out many applications of the projection methods. We observe that the projection method cannot be applied for the generalized nonlinear mixed quasivariational inequalities due to the presence of the nonlinear bifunction. The fact motivated many authors to develop the auxiliary principle technique to study the existence and iterative algorithm of solutions for various nonlinear mixed variational inequalities, e.g. see, Glowinske et al [4], Harker and Pang [5], Ding [3]. Recently, Ding [3] introduced a new class of predictor-corrector iterative algorithms for solving generalized mixed variational-like inequalities by applying the auxiliary principle technique.

Inspired and motivated by going on in this field, in this paper we use the auxiliary principle technique to suggest a class of predictor-corrector methods for solving generalized nonlinear mixed quasivariational inequalities. The convergence of the proposed methods only requires the continuity and the g-partially relaxed strong monotonicity of mappings. The results presented in this paper extend, improve and unify the corresponding results in this area.

2. Preliminaries

Let H be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, I stand for the identity mapping on H, $C(H)$ denote the families of nonempty compact subsets of H. Let $T, A : H \to C(H)$ be set-valued mappings. Let $N : H \times H \to H$ and $g : H \to H$ be single-valued mappings, and $\varphi : H \times H \to (-\infty, +\infty)$ be a continuous bifunction. We consider the generalized nonlinear mixed quasivariational inequality problem:

Find $x \in H, u \in T(x)$ and $v \in A(x)$ such that

$$\langle N(u,v), g(y) - g(x)\rangle + \varphi(g(y), g(x)) - \varphi(g(x), g(x)) \geq 0, \quad \forall g(y) \in H. \tag{2.1}$$

For a suitable and appropriate choice of the operators $T, A, g.N, \varphi$ and the space H, one can obtain various classes of variational inequalities and complementarity problems as special cases of problem (2.1). For detail, see [3], [6], [7], [9], [8], [10], [11] and the references therein. We also need the following concepts.

Definition 2.1. Let $T, A : H \to C(H)$ be set-valued mappings, $N : H \times H \to H$ and $g : H \to H$ be single-valued mappings.

(1) $N(\cdot,\cdot)$ is said to be *g-partially relaxed strongly monotone* in the first argument with respect to T if there exists a constant $\alpha > 0$ such that

$$\langle N(u,\cdot) - N(v,\cdot), g(z) - g(y)\rangle \\ \geq -\alpha\|g(x) - g(z)\|^2, \quad \forall x, y, z \in H,\ u \in T(x),\ v \in T(y).$$

(2) T is said to be *D-continuous* at $x_0 \in H$ if for each $\epsilon > 0$, there a neighborhood $N(x_0)$ of x_0 such that

$$D(T(x), T(x_0)) \leq \epsilon, \quad \forall x \in N(x_0),$$

where $D(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$.

Similarly, we can define the g-partially relaxed strong monotonicity of $N(\cdot,\cdot)$ in the second argument with respect to A.

If T, A are single-valued mappings, $N(Tx, Ax) = Tx$ and $g = I$, then the concept in (1) reduces to the concept of the partially relaxed monotonicity of Verma [12].

Definition 2.2. $\varphi(\cdot.\cdot) : H \times H \to (-\infty, +\infty]$ is said to be *skew-symmetric* if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0, \quad \forall u, v \in H.$$

Note that if $\varphi(\cdot,\cdot)$ is a bilinear function then $\varphi(\cdot,\cdot)$ is nonnegative.

3. Main Results

In this section, we suggest a new predictor-corrector iterative algorithm for solving problem (2.1) by using the auxiliary principle technique of Glowinski et al [4] as developed by Ding [3]. The convergence of the iterative sequence generated by the algorithm is proved.

For given $x \in H$, $u \in T(x)$ and $v \in A(x)$, we consider the following auxiliary variational inequality problem: find $w \in H$ such that

$$\begin{aligned}&\langle \rho N(u,v) + g(w) - g(x), g(y) - g(w)\rangle \\ &+\rho\varphi(g(y), g(w)) - \rho\varphi(g(w), g(w)) \geq 0, \quad \forall g(y) \in H,\end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant. We observe that if $w = x$, $u \in T(w)$ and $v \in A(w)$, then (w, u, v) is a solution of problem (2.1). By the observation, we can suggest the following predictor-corrector algorithm for solving problem (2.1).

Algorithm 3.1. For given $x_0 \in H$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute the approximate solution (x_n, u_n, v_n) of problem (2.1) by the following iterative schemes:

$$\begin{aligned}&\langle \mu N(u_n,v_n) + g(y_n) - g(x_n), g(y) - g(y_n)\rangle \\ &+\mu\varphi(g(y), g(y_n)) - \mu\varphi(g(y_n), g(y_n)) \geq 0, \quad \forall g(y) \in H,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\langle \beta N(c_n,d_n) + g(z_n) - g(y_n), g(y) - g(z_n)\rangle \\ &+\beta\varphi(g(y), g(z_n)) - \beta\varphi(g(z_n), g(z_n)) \geq 0, \quad \forall g(y) \in H,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\langle \rho N(e_n,f_n) + g(x_{n+1}) - g(z_n), g(y) - g(x_{n+1})\rangle \\ &+\rho\varphi(g(y), g(x_{n+1})) - \rho\varphi(g(x_{n+1}), g(x_{n+1})) \geq 0, \quad \forall g(y) \in H,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&u_n \in T(x_n), \quad \|u_{n+1} - u_n\| \leq D(T(x_{n+}), T(x_n)), \\ &v_n \in A(x_n), \quad \|v_{n+1} - v_n\| \leq D(A(x_{n+1}), A(x_n)), \\ &c_n \in T(y_n), \quad \|c_{n+1} - c_n\| \leq D(T(y_{n+1}), T(y_n)), \\ &d_n \in A(y_n), \quad \|d_{n+1} - d_n\| \leq D(A(y_{n+1}), A(y_n)), \\ &e_n \in T(z_n), \quad \|e_{n+1} - e_n\| \leq D(T(z_{n+1}), T(z_n)), \\ &f_n \in A(z_n), \quad \|f_{n+1} - f_n\| \leq D(A(z_{n+1}), A(z_n)),\end{aligned} \tag{3.5}$$

for all $n \geq 0$, where $\mu > 0$, $\beta > 0$, $\rho > 0$ are constants, and D is the Hausdorff metric on $C(H)$.

Remark 3.1. Algorithm 3.1 improves and generalizes Algorithms 3.1-3.3 of Ding [3].

Lemma 3.1. *Let (x, u, v) be a exact solution of problem (2.1) and x_n, u_n and v_n be the sequences of approximate solutions of problem (2.1) generated by Algorithm 3.1. If $N(\cdot,\cdot)$ is g-partially relaxed strongly monotone in the first* *and second arguments with respect to T and A with constants $\alpha > 0$ and $\gamma > 0$, respectively, and the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric, then*

$$\|g(x_{n+1}) - g(x)\|^2 \le \|g(x_n) - g(x)\|^2 - (1 - 2\rho(\alpha+\gamma))\|g(x_{n+1}) - g(z_n)\|^2 , \quad (3.6)$$

for any $n \ge 0$ and

$$\|g(z_n) - g(x)\|^2 \le \|g(z_{n-1}) - g(x)\|^2 - (1 - 2\beta(\alpha+\gamma))\|g(z_n) - g(y_n)\|^2, \quad (3.7)$$

$$\|g(y_n) - g(x)\|^2 \le \|g(y_{n-1}) - g(x)\|^2 - (1 - 2\mu(\alpha+\gamma))\|g(y_n) - g(x_n)\|^2 , \quad (3.8)$$

for all $n \ge 1$, where $0 < \rho, \beta, \mu < 1/(2(\alpha+\gamma))$.

Proof. Let (x, u, v) be a solution of problem (2.1), then $u \in T(x)$, $v \in A(x)$ and, for all $g(y) \in H$,

$$\langle \mu N(u,v), g(y) - g(x)\rangle + \mu\varphi(g(y), g(x)) - \mu\varphi(g(x), g(x)) \ge 0, \quad (3.9)$$

$$\langle \beta N(u,v), g(y) - g(x)\rangle + \beta\varphi(g(y), g(x)) - \beta\varphi(g(x), g(x)) \ge 0, \quad (3.10)$$

$$\langle \rho N(u,v), g(y) - g(x)\rangle + \rho\varphi(g(y), g(x)) - \rho\varphi(g(x), g(x)) \ge 0, \quad (3.11)$$

where $\mu > 0$, $\beta > 0$, $\rho > 0$ are constants.

Taking $y = x_{n+1}$ in (3.11) and $y = x$ in (3.4), we have

$$\langle \rho(N(u,v), g(x_{n+1}) - g(x)\rangle + \rho\varphi(g(x_{n+1}), g(x)) - \rho\varphi(g(x), g(x)) \ge 0, \quad (3.12)$$

$$\langle \rho N(e_n, f_n) + g(x_{n+1}) - g(z_n), g(x) - g(x_{n+1})\rangle + \rho\varphi(g(x), g(x_{n+1})) - \rho\varphi(g(x_{n+1}), g(x_{n+1})) \ge 0. \quad (3.13)$$

Adding (3.12) and (3.13), we get that

$$\begin{aligned} & \langle g(x_{n+1}) - g(z_n), g(x) - g(x_{n+1})\rangle \\ \ge\ & \rho\langle N(e_n, f_n) - N(u,v), g(x_{n+1}) - g(x)\rangle \\ & +\rho(\varphi(g(x), g(x)) - \varphi(g(x_{n+1}), g(x)) - \varphi(g(x), g(x_{n+1})) \\ & +\varphi(g(x_{n+1}), g(x_{n+1}))) \\ \ge\ & \rho\langle N(e_n, f_n) - N(u, f_n), g(x_{n+1}) - g(x)\rangle \\ & +\rho\langle N(u, f_n) - N(u,v), g(x_{n+1}) - g(x)\rangle \\ \ge\ & -\rho(\alpha+\gamma)\|g(x_{n+1}) - g(z_n)\|^2, \end{aligned} \quad (3.14)$$

where we have used the assumption that $N(\cdot,\cdot)$ is g-partially relaxed strongly monotone in the first and second arguments with respect to T and A with constants $\alpha > 0$ and $\gamma > 0$, respectively, and $\varphi(\cdot,\cdot)$ is skew-symmetric. Since

$$\begin{aligned}\|g(x)-g(z_n)\|^2 &= \|g(x)-g(x_{n+1})+g(x_{n+1})-g(z_n)\|^2\\ &= \|g(x_{n+1})-g(x)\|^2+\|g(x_{n+1})-g(z_n)\|^2\\ &\quad +2\langle g(x_{n+1})-g(z_n), g(x)-g(x_{n+1})\rangle.\end{aligned}$$

It follows from (3.14) that

$$\begin{aligned}&\langle g(x_{n+1})-g(z_n), g(x)-g(x_{n+1})\rangle\\ =\ & \frac{1}{2}[\|g(x)-g(z_n)\|^2-\|g(x_{n+1})-g(x)\|^2-\|g(x_{n+1})-g(z_n)\|^2]\\ \geq\ & -\rho(\alpha+\gamma)\|g(x_{n+1})-g(z_n)\|^2.\end{aligned}$$

Therefore, we get that for $\rho < 1/(2(\alpha+\gamma))$,

$$\begin{aligned}&\|g(x_{n+1})-g(x)\|^2\\ \leq\ & \|g(z_n)-g(x)\|^2-(1-2\rho(\alpha+\gamma))\|g(x_{n+1})-g(z_n)\|^2\\ \leq\ & \|g(z_n)-g(x)\|^2.\end{aligned} \tag{3.15}$$

Taking $y = z_n$ in (3.10), $y = x$ in (3.3) and $y = y_n$ in (3.9), $y = x$ in (3.2), respectively, observe that $0 < \beta,\ \mu < 1/(2(\alpha+\gamma))$, similarly as the above discussion, we have

$$\begin{aligned}&\|g(z_n)-g(x)\|^2\\ \leq\ & \|g(y_n)-g(x)\|^2-(1-2\beta(\alpha+\gamma))\|g(z_n)-g(y_n)\|^2\\ \leq\ & \|g(y_n)-g(x)\|^2,\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\|g(y_n)-g(x)\|^2\\ \leq\ & \|g(x_n)-g(x)\|^2-(1-2\mu(\alpha+\gamma))\|g(y_n)-g(x_n)\|^2\\ \leq\ & \|g(x_n)-g(x)\|^2.\end{aligned} \tag{3.17}$$

It follows from (3.15)-(3.17) that the conclusions (3.6)-(3.8) hold. This completes the proof. □

Theorem 3.1. *Let H be a finite dimensional Hilbert space, $T, A : H \to C(H)$ be D-continuous set-valued mappings and $N : H \times H \to H$ and $g : H \to H$ be continuous single-valued mappings such that g is invertible. Let $\varphi : H \times H \to (-\infty, +\infty)$ be a continuous bifunction such that φ is skew-symmetric. Suppose that N is g-partially relaxed strongly monotone in the first and the second arguments with respect to T and A with constants $\alpha > 0$*

and $\gamma > 0$, respectively, and that the solution set of problem (2.1) *is nonempty. Then for any given $x_0 \in H$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$ the iterative sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ defined by Algorithm 3.1 with $0 < \rho, \beta, \mu < 1/(2(\alpha + \gamma))$ converge strongly to a solution (x, u, v) of problem* (2.1).

Proof. Let (x, u, v) be a solution of problem (2.1). It follows from Lemma 3.1 that the sequences $\{\|g(x_{n+1}) - g(x)\|\}$, $\{\|g(z_n) - g(x)\|\}$ and $\{\|g(y_n) - g(x)\|\}$ are nonincreasing and consequently $\{x_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\rho(\alpha + \gamma))\|g(x_{n+1}) - g(z_n)\|^2 \leq \|g(x_0) - g(x)\|^2,$$
$$\sum_{n=1}^{\infty}(1 - 2\beta(\alpha + \gamma))\|g(z_n) - g(y_n)\|^2 \leq \|g(z_0) - g(x)\|^2,$$
$$\sum_{n=1}^{\infty}(1 - 2\mu(\alpha + \gamma))\|g(y_n) - g(x_n)\|^2 \leq \|g(y_0) - g(x)\|^2.$$

These inequalities imply $\|g(x_{n+1}) - g(z_n)\| \to 0$, $\|g(z_n) - g(y_n)\| \to 0$ and $\|g(y_n) - g(x_n)\| \to 0$ as $n \to \infty$. Therefore, we have

$$\begin{aligned} & \|g(x_{n+1}) - g(x_n)\| \\ \leq\ & \|g(x_{n+1}) - g(z_n)\| + \|g(z_n) - g(y_n)\| + \|g(y_n) - g(x_n)\| \\ \to\ & 0 \quad \text{as } n \to \infty. \end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \bar{x}$ as $i \to \infty$ and hence we have $g(x_{n_i}) \to g(\bar{x})$ and $g(y_{n_i}) \to g(\bar{x})$ as $i \to \infty$. Since T and A are D-continuous on H, by Proposition 1.5.2 of Aubin and Cellina [1], p. 66, T and A are both upper semicontinuous on H with compact values. Note that $u_n \in T(x_n)$ and $v_n \in A(x_n)$ for all $n \geq 0$. It follows from Proposition 11.11 of Border [2], p. 57 that there exist subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ and subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ such that $u_{n_{i_j}} \to \bar{u}$, $v_{n_{i_j}} \to \bar{v}$ as $j \to \infty$, where $\bar{u} \in T(x)$ and $\bar{v} \in A(x)$, respectively. By (3.2), we have

$$\begin{aligned} & \langle \mu N(u_{n_{i_j}}, v_{n_{i_j}}) + g(y_{n_{i_j}}) - g(x_{n_{i_j}}), g(y) - g(y_{n_{i_j}}) \rangle \\ & + \mu\varphi(g(y), g(x_{n_{i_j}})) - \mu\varphi(g(x_{n_{i_j}}), g(x_{n_{i_j}})) \geq 0, \quad \forall g(y) \in H. \end{aligned} \tag{3.18}$$

By the continuity of $N(\cdot, \cdot)$, g and φ, letting $j \to \infty$ in (3.25), we obtain that

$$\langle N(\bar{u}, \bar{v}), g(y) - g(\bar{x}) \rangle + \varphi(g(y), g(\bar{x})) - \varphi(g(\bar{x}), g(\bar{x})) \geq 0, \quad \forall g(y) \in H,$$

that is, $(\bar{x}, \bar{u}, \bar{v})$ is a solution of problem (2.1). Lemma 3.1 ensure that

$$\|g(x_{n+1}) - g(\bar{x})\| \leq \|g(x_n) - g(\bar{x})\|, \quad \forall n \geq 0,$$

which implies that $g(x_n) \to g(\bar{x})$ as $n \to \infty$. Since g is invertible, it follows that $x_n \to \bar{x}$ as $n \to \infty$. Since T and A are D-continuous on H, by (3.5), we have

$$\|u_n - u_{n+1}\| \leq D(T(x_n), T(x_{n+1})) \to 0 \quad \text{as } n \to \infty.$$

It follows that for any $n > 0$

$$\begin{aligned} \|u_n - \bar{u}\| \quad \leq \quad & \|u_n - u_{n+1}\| + \|u_{n+1} - u_{n+2}\| + \cdots \\ & \|u_{n_{i_j}-1} - u_{n_{i_j}}\| + \|u_{n_{i_j}} - \bar{u}\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

That is, $u_n \to \bar{u}$ as $n \to \infty$. Similarly, we can prove that $v_n \to \bar{v}$ as $n \to \infty$. This completes the proof. □

Remark 3.2. Theorem 3.1 extends and generalizes Theorem 3.1 of Ding [3].

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