THE HANKEL TRANSFORM OF $k$-TH DERIVATIVE OF DIRAC DELTA IN $u(x_1, \ldots, x_n)$

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Abstract: In this article we obtain the Hankel Transform of $k$-th derivative of Dirac delta in $u(x_1, \ldots, x_n)$, where $u \in C^\infty(R^n)$ be such that $(n-1)$ dimensional manifold $u(x_1, \ldots, x_n) = 0$ has no critical point. As first consequence we obtain the distributional Hankel Transform of $\delta^{(k)}(P(x) + m^2)$ which was published in ([2], p. 113, formula (I.2.6)), where $P(x) + m^2$ is defined by the equation (33). As second consequence we give a sense to distributional Hankel Transform of $\delta^{(k)}(u(x_1, \ldots, x_n) - t)$.

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1. Introduction

Let $U(t) \in S'_R^+$, where $S'_R^+$ is the dual of $S_R^+$, the space of functions $f \in S$ defined in the positive half line $R^+ \{t \mid t > 0\}$. Here $S$ is the Schwartz space of functions([5], p. 233). Then the Hankel Transform of $U(t)$ will be, by definition, the distribution $V(s) \in S'_R^+$ defined by the formula

$$\langle H(U(t)), \varphi(s) \rangle = \langle U(t), H(\varphi(s)) \rangle,$$

(1)

for every $\varphi \in S'_R^+$ (c.f. [6], p. 26, equation (1.5.6)).

By the Hankel Transform of the function $f(t)$ we mean the function $g(s)$, $0 \leq s < \infty$, defined by the formula

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\[ g(s) = (H \{ f(t) \}) = \frac{1}{2} \int_0^\infty f(t) t^{n-2} R_{\frac{n-2}{2}}(\sqrt{ts}) dt, \quad (2) \]

where
\[ R_\gamma(x) = \frac{J_\gamma(x)}{x^\gamma}, \quad (3) \]

and
\[ J_\gamma(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\frac{x}{2})^{\gamma+2\nu}}{\nu! \Gamma(\gamma + \nu + 1)}. \quad (4) \]

Similarly, let \( U(t) \in S'_R \). The Hankel Transform of \( U(t) \) will be, by definition, the distribution \( V(s) \in S'_R \), defined by the formula
\[ \langle H(U(t)), \varphi(s) \rangle = \langle U(t), H(\varphi(s)) \rangle, \quad (5) \]

for every \( \varphi \in S_R \). Here \( S'_R \) designates the dual of \( S_R \). By \( S_R \) we designate the space of functions \( f \in S \) defined in the negative half line \( R^- = \{ t \mid t < 0 \} \). By the Hankel Transform of the function \( \varphi(s) \) we mean the function \( h(t), \) \( -\infty < t < 0, \) defined by the formula
\[ h(t) = (H \{ \varphi(s) \}) = (-1)^{\frac{1}{2}} \int_{-\infty}^0 \varphi(s) s^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st}) ds. \quad (6) \]

On the other hand, from [2], pp. 111-112, equation (I.1.2) and equation (I.1.3) we have
\[ H \{ \delta^{(m)}(t) \} = \frac{1}{2m+1} R_{\frac{n-2}{2}+m}(0) s^{\frac{n-2}{2}+m}, \quad (7) \]

where
\[ R_{\frac{n-2}{2}+m}(0) = \frac{\Gamma\left(\frac{n+2m-1}{2}\right)}{2^{\frac{n-2}{2}+m} \Gamma\left(\frac{n-2}{2} + m + \frac{1}{2}\right) \Gamma\left(\frac{n+2m}{2}\right)}, \quad (8) \]

\[ \Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx \quad (9) \]

and \( \delta^{(m)}(t) \) is the \( m \)-th derivative of the Dirac measure.

Let \( \phi_t \) denote a distribution of one variable \( t \). Let \( u \in C^\infty(R^n) \) be such that \( (n-1) \)-dimensional manifold \( u(x_1, \ldots, x_n) = 0 \) has no critical point. By \( \phi_{u(x)} \) Leray (c.f. [4], p. 102) designates the distribution defined on \( R^n \) by
\[ \langle \phi_{u(x)}, \varphi(x) \rangle = \langle \phi_t, \psi(t) \rangle \quad ([4], p. 102), \quad (10) \]
where
\[ \psi(t) = \int_{u(x)=t} \varphi(x) w_u(x, dx) \] (11)
and \( \varphi \in C_0(R^n) \) is the set of infinitely differentiable functions with compact support and \( w_u \) is a \((n-1)\) dimensional exterior differential form on \( u \) defined as follows:
\[ du \wedge dw = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \] (12)
and the orientation of the manifold \( u(x) = t \) is such that \( w_u(x, dx) > 0 \).

On the other hand from [3], p. 230, formula (6), we have:
\[ \langle \delta^{(k)}(G(x_1, x_2, \ldots, x_n), \varphi_1(x_1, x_2, \ldots, x_n)) \rangle = (-1)^k \int_{G(x)=0} w_k(\varphi), \] (13)
k = 0, 1, 2, \ldots, where \( x = (x_1, x_2, \ldots, x_n) \), \( G(x_1, x_2, \ldots, x_n) \) is such an infinite differentiable function that
\[ \text{grad} G = \left( \frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_n} \right) \neq 0, \] (14)
\[ w_k(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{ D(u) \varphi_1(u_1, u_2, \ldots, u_n) \right\} du_2 \ldots du_n, \] (15)
w_0 = \varphi \cdot w, \] (16)
\( u_1 = G(x_1, x_2, \ldots, x_n), \]
\( u_2 = x_2, \]
\[ \vdots \]
\( u_n = x_n, \] (17)
\( \varphi_1 \) is defined by equation (11) and
\[ D(u) = \left[ D(u) \right]^{-1} = \frac{1}{\partial G/\partial x_1}, \] (18)
with
\[ \frac{\partial G}{\partial x_1} > 0. \] (19)

Otherwise, from [3], p. 211, formula (8), \( \delta^{(k)}(G(x_1, x_2, \ldots, x_n)) \) can be written as
\[ \langle \delta^{(k)}(G(x), \varphi) \rangle = (-1)^k \int_{G=0} f_{u_1}^{(k)}(0, u_2, \ldots, u_n) du_2 \ldots du_n = \]
\[-1 \int_{G=0} \left[ \frac{\partial^k}{\partial u_1^k} f^{(k)}(0, u_2, \ldots, u_n) \right] \, du_2 \ldots du_n, \tag{20} \]

where

\[f(u_1, u_2, \ldots, u_n) = \varphi_1(u_1, u_2, \ldots, u_n) D \left( \frac{x}{u} \right), \tag{21}\]

\[\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(x_1, x_2, \ldots, x_n), \tag{22}\]

and \(D \left( \frac{x}{u} \right)\) is defined by (18).

In this article we obtain the Hankel Transform of \(k\)-th derivative of Dirac delta in \(u(x_1, \ldots, x_n)\), where \(u \in C^\infty(R^n)\) be such that \((n-1)\) dimensional manifold \(u(x_1, \ldots, x_n) = 0\) has no critical point. As consequence we obtain the distributional Hankel Transform of \(\delta^{(k)}(P(x) + m^2)\) which was published in [2], p. 113, formula (I.2.6), where \(P(x) + m^2\) is defined by the equation (33).

### 2. The Hankel Transform of \(\delta^{(k)}(u(x_1, \ldots, x_n))\)

In this paragraph we give a sense to Hankel Transform of \(\delta^{(k)}(u(x_1, \ldots, x_n))\), where \(u \in C^\infty(R^n)\) be such that \((n-1)\) dimensional manifold \(u(x_1, \ldots, x_n) = 0\) has no critical point.

To obtain our results we need the following formula

\[\langle H \{ \phi(U(x)) \}, \varphi(x) \rangle = \langle H \{ V(t) \}, \psi(s) \rangle \tag{23}\]

(see [2], p. 2, formula (11)), where

\[V(t) = \phi(U), \tag{24}\]

and

\[\psi(s) = \int_{u=s} \varphi w(x, dx). \tag{25}\]

**Theorem 1.** Let \(\phi_t\) denote a distribution of one variable \(t\). Let \(u \in C^\infty(R^n)\) be such that \((n-1)\) dimensional manifold \(u(x_1, \ldots, x_n) = 0\) has no critical point. Then the following formula is valid

\[H \left\{ \delta^{(k)}(u(x_1, \ldots, x_n)) \right\} = B_{k,n}(u(y_1, \ldots, y_n) \frac{n-2}{2} + k, \tag{26}\]

where \(\delta^{(k)}(u(x_1, \ldots, x_n))\) is defined by the equation (20),

\[B_{k,n} = \frac{1}{2^{2k+n} \Gamma \left( \frac{n}{2} + k \right)}, \tag{27}\]

\(u(y_1, \ldots, y_n) \in C^\infty(R^n)\) be such that \((n-1)\) dimensional manifold \(u(y_1, \ldots, y_n) = 0\) has no critical point.
Proof. From (23), (24) and (25) we have
\[
\left\langle H \left\{ \delta^{(k)}(u(x_1, \ldots, x_n)) \right\}, \varphi(y_1, \ldots, y_n) \right\rangle = \left\langle H \left\{ \delta^{(k)}(t) \right\}, \psi(s) \right\rangle, \tag{28}
\]
where \( \psi(s) \) is defined by the equation (22).

Now from (28) and using (7) and (8), we have
\[
\left\langle H \left\{ \delta^{(k)}(u(x_1, \ldots, x_n)) \right\}, \varphi(y_1, \ldots, y_n) \right\rangle = B_{k,n} \left\langle s^{\frac{n-2}{2}+k}, \psi(s) \right\rangle, \tag{29}
\]
where
\[
B_{k,n} = \frac{1}{2^{2k+n} \Gamma(n/2+k)} \tag{30}
\]
From (29) and considering the Leray formula (see 10) we have
\[
\left\langle H \left\{ \delta^{(k)}(u(x_1, \ldots, x_n)) \right\}, \varphi(y) \right\rangle = B_{k,n} \left\langle (u(y))^{\frac{n-2}{2}+k}, \varphi(y) \right\rangle, \tag{31}
\]
where \((y) = (y_1, \ldots, y_n)\). From (31) we obtain the formula
\[
H \left\{ \delta^{(k)}(u(x_1, \ldots, x_n)) \right\} = B_{k,n} (u(y))^{\frac{n-2}{2}+k}, \tag{32}
\]
where \( B_{k,n} \) is defined by (30).

From (32) we conclude the theorem proof.  \( \Box \)

We shall consider this special case
\[
u(x) = u(x_1, \ldots, x_n) = P(x) + m^2 = P(x_1, \ldots, x_n) + m^2. \tag{33}
\]
Here,
\[
P(x) = P(x_1, \ldots, x_n) = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2, \tag{34}
\]
\( \mu + \nu = n \) dimension of the space and \( m > 0 \).

Therefore, from (26), we have
\[
H \left\{ \delta^{(k)}(P(x) + m^2) \right\} = B_{k,n} (Q(y) + m^2)^{\frac{n-2}{2}+k}, \tag{35}
\]
where
\[
Q(y) + m^2 = x_1^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - \cdots - x_{\mu+\nu}^2 + m^2 \tag{36}
\]
and \( B_{k,n} \) is defined by the equation (30).

The formula (35) appears in (see [2]), p. 113, formula (I.2.6).
3. The Hankel Transform of $\delta^{(k)}(u(x_1, \ldots, x_n) - t)$

We know that the following formula is valid

$$\delta^{(k)}(u(x_1, \ldots, x_n) - t) = \sum_{j \geq o} \frac{(-1)^j}{j!} \delta^{(j+k)}(u(x_1, \ldots, x_n)) t^j,$$  \hspace{1cm} (37)

which appear in [1], where $t$ is a real number and $\delta^{(k)}(G(x_1, x_2, \ldots, x_n)$ is defined by the formula (13).

Let $a_r$ the sequence defined by

$$a_r = \sum_{j=o}^r \frac{(-1)^j}{j!} \delta^{(j+k)}(u(x_1, \ldots, x_n)) t^j,$$  \hspace{1cm} (38)

using the formulae (23) and (26) we have

$$H(a_r) = \sum_{j=o}^r \frac{(-1)^j}{j!} B_{k+j,n}(u(y_1, \ldots, y_n)) t^j,$$  \hspace{1cm} (39)

where the coefficient $B_{k+j,n}$ is defined by (27).

Now taking into account that the sequence $a_r$ converge uniformly in every compact $K \subset \mathbb{R}^n$ and by the continuity of the Hankel Transform ([7], p. 142), the sequence(39) converge uniformly in every compact $K \subset \mathbb{R}^n$, we conclude that the Hankel Transform of

$$\delta^{(k)}(u(x_1, \ldots, x_n) - t)$$

is given by the following formula

$$H \left\{ \delta^{(k)}(u(x_1, \ldots, x_n) - t) \right\} = \sum_{j \geq o} \frac{(-1)^j}{j!} B_{k+j,n}(u(x_1, \ldots, x_n)) \frac{n^2}{2} + k + j t^j,$$  \hspace{1cm} (40)

where $B_{k+j,n}$ is defined by the formula (27).

We observed that putting $t = 0$ in (40) we obtain the formula (26).
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References

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