REORDERINGS OF SERIES IN BANACH SPACES AND SOME PROBLEMS IN NUMBER THEORY

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Abstract: We review some results in reorderings of series in Banach spaces that have some applications in analytic number theory.

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1. Introduction

The purpose of this note is to point out some problems in number theory, dealing mainly with properties of the Riemann Zeta Function \( \zeta \), for which the theory or reorderings of series in Banach spaces are (or could be) useful. First we review briefly some results of this theory and later we mention (probable) applications to number theory.

Let us start with some definitions. If \( \sum_{n \geq 1} x_n \) is a series in a topological vector space \( E \), \( \pi : \mathbb{N} \to \mathbb{N} \) is a bijection and the series \( \sum_{n \geq 1} x_{\pi(n)} \) is convergent in \( E \), we called this latter series a convergent reordering of the series \( \sum_{n \geq 1} x_n \). The set of all sums of all convergent reorderings of the series \( \sum_{n \geq 1} x_n \) will be denoted by \( \text{CR}(\sum_{n \geq 1} x_n) \). Evidently this set could be empty. When \( E = \mathbb{R} \), B. Riemann...
(1854) proved that \( \text{CR}(\sum_{n \geq 1} x_n) = \mathbb{R} \) if \( \sum_{n \geq 1} x_n \) is conditionally convergent and G. Dirichlet (1837) proved that \( \text{CR}(\sum_{n \geq 1} x_n) \) is a singleton if \( \sum_{n \geq 1} x_n \) is absolutely convergent. When \( E = \mathbb{C} = \mathbb{R}^2 \), P. Levy (1905) proved that if \( \text{CR}(\sum_{n \geq 1} x_n) \neq \emptyset \), then it is an affine set, that is, the translate of a linear subspace. This result was extended to all of \( \mathbb{R}^k \) by E. Steinitz (1913): If \( \text{CR}(\sum_{n \geq 1} x_n) \neq \emptyset \), \( x_n \in \mathbb{R}^k \), \( n \geq 1 \), then \( \text{CR}(\sum_{n \geq 1} x_n) \) is an affine set in \( \mathbb{R}^k \). This last result is known as the theorem of Lévy-Steinitz.

V. M. Kadets (1986) proved that in any infinite dimensional Banach space there is a series \( \sum_{n \geq 1} x_n \) such that \( \text{CR}(\sum_{n \geq 1} x_n) \neq \emptyset \) and it is not convex. More specifically, M. I. Kadets and K. Wóźniakowsky (1989), P. A. Kornilov (1988) and P. Enflo (unpublished) proved, independently, that in any infinite dimensional Banach space there is a series \( \sum_{n \geq 1} x_n \) such that \( \text{CR}(\sum_{n \geq 1} x_n) \) is a two point set.

W. Banaszczyk (1990, 1993) has shown that a Fréchet space is a nuclear if and only if the Lévy-Steinitz Theorem holds in it.

For the results quoted above, see Diestel et al [6].

The following results on conditionally convergent series in a real Hilbert space appears in Karatsuba et al [9]:

**Theorem 1.** If \( \sum_{n \geq 1} x_n \) is a series in a real Hilbert space \( H \), such that \( \sum_{n \geq 1} \|x_n\|^2 < \infty \) and \( \forall z \in H, \|z\| = 1 \), \( \text{CR}(\sum_{n \geq 1} \langle x_n, z \rangle) = \mathbb{R} \), then \( \text{CR}(\sum_{n \geq 1} x_n) = H \).

If \( T \) is a compact operator in the separable Hilbert space \( H \) and \( \varphi = \{\varphi_i\}_{i \in \mathbb{N}} \) is an orthonormal basis in \( H \), we say that \( \varphi \in \text{Dom}(\text{tr}T) \) if the series \( \sum_{i \geq 1} \langle T\varphi_i, \varphi_i \rangle \) converges. If \( \varphi \in \text{Dom}(\text{tr}T) \), we denote the complex number \( \sum_{i \geq 1} \langle T\varphi_i, \varphi_i \rangle \) by \( \text{tr}_\varphi T \). This correspondence defines a function from \( \text{Dom}(\text{tr}T) \) into \( \mathbb{C} \). The range of this function will be denoted by \( \text{R}(\text{tr}T) \). It is possible to have \( \text{Dom}(\text{tr}T) = \emptyset \) and therefore also \( \text{R}(\text{tr}T) = \emptyset \).

The following theorem was proven by A. Ben-Artzi (1984), Goldberg et al
Theorem. Let $T \in B(H)$ be compact, $H$ a separable Hilbert space such that $\text{Dom}(\text{tr}T) \neq \emptyset$. Then:

1. $R(\text{tr}T)$ is an affine set of $\mathbb{C}$.

2. There exists an orthonormal basis $\varphi = \{\varphi_i\}_{i \geq 1}$ such that $R(\text{tr}T) = \text{CR}\left(\sum_{i \geq 1} \langle T\varphi_i, \varphi_i \rangle\right)$.

3. $R(\text{tr}T)$ is a straight line in $\mathbb{C}$ if and only if $T$ can be represented as $T = \mu A + R$, where $\mu$ is a trace class operator, $\mu \in \mathbb{C} \setminus \{0\}$, $A = A^*$ and neither $A_+^+$ nor $A_-^-$ is of trace class.

Finally, the author of this note proved in 1996, Alcántara-Bode [2], the following result about reorderings of harmonic like divergent series.

Theorem 2. If $f : \mathbb{N} \to \mathbb{C}$ is a function such that $\sum_{n \geq 1} \left| f(n) - \frac{1}{n} \right| < \infty$ and $\alpha \in [0, 1] \setminus \mathbb{Q}$, then

$$\sum_{n \geq 1} \left\{ f\left(\left\lfloor \frac{n}{\alpha} \right\rfloor\right) + f\left(\left\lfloor \frac{n}{1-\alpha} \right\rfloor\right) - f(n) \right\}$$

$$= -\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha),$$

where $\lfloor x \rfloor$ denotes the integer part of $x$.

2. Applications to Number Theory

We now mention some problems in number theory, some still unsolved, for which the above results are or could be useful.

If $\rho(x) = x - \lfloor x \rfloor$ is the fractionary part function, in 1955 A. Beurling, Donoghue [7], proved the following theorem.

Theorem. If

$$M = \{ f : f(x) = \sum_{k=1}^{N} a_k \rho\left(\frac{\theta_k}{x}\right), \sum_{k=1}^{N} a_k \theta_k = 0, 0 < \theta_k \leq 1, a_k \in \mathbb{C},$$

$$1 \leq k \leq N, N \geq 2 \},$$

then the Riemann Hypothesis (R.H.) holds if and only if $\overline{M} = L^2(0, 1)$. Moreover, $\overline{M} = L^2(0, 1)$ if and only if $1 \in \overline{M}$. 

If $\mu$ is the Möbius function, E. Meissel (1854), Narkiewicz [10], has proven that if $x \geq 1$ then \[ \sum_{n \geq 1} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1, \] and H. von Mangoldt (1897), Narkiewicz [10], has shown that \[ \sum_{n \geq 1} \frac{\mu(n)}{n} = 0. \] Combining these last two equations we get that pointwise it holds that
\[
\sum_{n \geq 1} \mu(n) \left\{ \rho \left( \frac{\theta}{nx} \right) - \frac{1}{n} \rho \left( \frac{\theta}{x} \right) \right\} = -\chi_{[0,\theta]}(x), \quad \forall \theta, x \in [0,1].
\] (1)

The partial sums of this series are in $M$, but the series does not converge strongly in $L^2(0,1)$ (R. Heath-Brown, private communication); but to establish R.H. weak convergence would suffice.

For any bijection $\pi$ of $\mathbb{N}$ it can be shown that
\[
\sum_{n \geq 1} \mu(\pi(n)) \left\{ \rho \left( \frac{\theta}{\pi(n)x} \right) - \frac{1}{\pi(n)} \rho \left( \frac{\theta}{x} \right) \right\} = \left\lfloor \frac{\theta}{x} \right\rfloor \sum_{n \geq 1} \frac{\mu(\pi(n))}{\pi(n)} - \chi_{[0,\theta]}(x), \quad \forall \theta, x \in [0,1].
\]

Since the series \[ \sum_{n \geq 1} \frac{\mu(n)}{n} \] is conditionally convergent, \[ \sum_{n \geq 1} \frac{\mu(\pi(n))}{\pi(n)} \] can take any real value or even be divergent. Therefore since \( g_{\theta} \notin L^2(0,1) \), where \( g_{\theta}(x) = \left\lfloor \frac{\theta}{x} \right\rfloor, \theta \in [0,1] \), a natural question to ask is: Are there any bijections $\pi$ of $\mathbb{N}$ such that \[ \sum_{n \geq 1} \frac{\mu(\pi(n))}{\pi(n)} = 0 \] and the series
\[
\sum_{n \geq 1} \mu(\pi(n)) \left\{ \rho \left( \frac{\theta}{\pi(n)x} \right) - \frac{1}{\pi(n)} \rho \left( \frac{\theta}{x} \right) \right\} = -\chi_{[0,\theta]}(x)
\] converges weakly in $L^2(0,1)$?

S.M. Voronin (1973), Karatsuba et al [9], has used Theorem 1 to prove the following differential independence result for the Riemann Zeta Function.

**Theorem.** If \( F(\zeta(s), \zeta'(s), \ldots, \zeta^{(N-1)}(s)) = 0 \) identically in \( s \in \mathbb{C} \), where \( F \) is a continuous complex function, then \( F \) is identically zero.
B. Bagchi (1982), Titchmarsh [11], has generalized this result proving that
\( \zeta \) does not satisfy a differential-difference equation: If \( h_1 < h_2 < \cdots < h_m \) are
real constants and

\[
\Phi(\zeta(s + h_1), \ldots, \zeta^{(n_1)}(s + h_1), \ldots, \zeta(s + h_m), \zeta^{(n_m)}(s + h_m)) = 0,
\]

where \( \Phi \) is a continuous complex function, then \( \Phi = 0 \).

In Alcántara-Bode [1], [4] the present author proved that if \( (A_\rho f)(\theta) = \int_0^1 \rho \left( \frac{\theta}{x} \right) f(x) dx \) is considered as an operator on \( L^2(0,1) \), then R.H. holds
if and only if ker \( A_\rho = \{0\} \). Among other things we proved that \( A_\rho \) is Hilbert-Schmidt but neither nuclear nor normal, and we determined its spectrum, its
(generalized) eigenvectors and its modified Fredholm determinant. If \( \{\lambda_n\}_{n \geq 1} \)
is the sequence of non-zero eigenvalues of \( A_\rho \), ordered in such a way that
\( |\lambda_n| \geq |\lambda_{n+1}|, \forall n \in \mathbb{N} \), where each eigenvalue is repeated a number of times
equal to its multiplicity, then \( |\lambda_n| \leq \frac{e}{n} \forall n \in \mathbb{N}, \sum_{n \geq 1} |\lambda_n| = \infty \) and \( \lambda_n \notin \mathbb{R} \) for
an infinite number of \( n \)’s. Two natural questions that arise are:

1) Is the series \( \sum_{n \geq 1} \lambda_n \) conditionally convergent? and if so, what alternative
of Lévy’s Theorem holds for it?

2) Which alternative of Ben-Artzi’s Theorem applies to \( A_\rho \)?

We have shown that \( \mathbb{R} \subset \mathbb{R}(\text{tr} A_\rho) \) which implies that either \( \mathbb{R}(\text{tr} A_\rho) = \mathbb{R} \) or \( \mathbb{C} \).

The Beurling function defined by

\[
J(\alpha) = \int_0^1 \rho \left( \frac{1}{x} \right) \rho \left( \frac{\alpha}{x} \right) dx, \alpha \in [0,1]
\]

appears in the Beurling approach to the R.H., Alcántara-Bode [3], Báez-Duarte
et al [5]. It has been shown in Báez-Duarte et al [5] that \( J \) is absolutely
continuous, it has a strict local maximum and it is no differentiable at every
rational point of \([0,1]\). This function obeys the functional equation

\[
-\frac{\alpha \ln \alpha}{2} - \frac{(1-\alpha) \ln (1-\alpha)}{2} = J(1) - J(\alpha) - J(1-\alpha) + \alpha
\]

\[
+ (1-\alpha)J \left( \frac{\alpha}{1-\alpha} \right), \forall \alpha \in [0,1/2], \quad (2)
\]
that relates the values of $J$ at 3 points. $J$ also obeys a functional equation that relates the values at 5 points.

For $J$ we have several expressions as an infinite series:

$$J(\alpha) = K(\alpha), \quad \forall \alpha \in [0, 1] \setminus \mathbb{Q},$$

$$J(\alpha) = K(\alpha) + \frac{p + q}{q} \left\{ \ln \Gamma \left( 1 - \frac{1}{p+q} \right) - \frac{\gamma}{p+q} \right\}$$

if $\alpha = \frac{p}{q} \in [0, 1] \cap \mathbb{Q}, p, q \in \mathbb{N}, (p, q) = 1$, where $\Gamma$ is the Gamma Function, $\gamma$ is the Euler constant and

$$K(\alpha) = \frac{\ln(1 + \alpha)}{2} + \frac{\alpha}{2} \ln \left( \frac{1 + \alpha}{\alpha} \right) - \alpha$$

$$- (1 + \alpha) \sum_{m \geq 1} \left\{ \ln \left( 1 - \frac{\rho(m\alpha)}{m(1 + \alpha)} \right) + \frac{\rho(m\alpha)}{m(1 + \alpha)} \right\}$$

$$- (1 + \alpha) \sum_{m \geq 1} \left\{ \ln \left( 1 - \frac{\rho\left(\frac{m}{\alpha}\right)}{m(1 + \alpha)} \right) + \frac{\rho\left(\frac{m}{\alpha}\right)}{m(1 + \alpha)} \right\}.$$

A formula that holds for all $\alpha$ in $[0, 1]$ is

$$J(\alpha) = -\frac{\alpha \ln \alpha}{2} - \sum_{n \geq 1} \left\{ \left( \sum_{k=1}^{\left\lfloor \frac{n}{\alpha} \right\rfloor} \frac{1}{k} \right) - \ln \left\lfloor \frac{n}{\alpha} \right\rfloor - \gamma - \frac{1}{2} \left\lfloor \frac{n}{\alpha} \right\rfloor \right\}$$

$$- \sum_{n \geq 1} \left\{ \ln \left( 1 - \frac{\rho\left(\frac{n}{\alpha}\right)}{\frac{n}{\alpha}} \right) + \frac{\rho\left(\frac{n}{\alpha}\right)}{\frac{n}{\alpha}} \right\} - \frac{1}{2} \sum_{n \geq 1} \frac{\rho\left(\frac{n}{\alpha}\right)}{\frac{n}{\alpha}}$$

$$+ \frac{\alpha}{2} \{ \ln(2\pi) - \gamma - 1 \}.$$  

These formulae for $J$ and the functional equation (2) are compatible by virtue of Theorem 2.

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References


