CUSPIDAL CURVES ON
THE SMOOTH QUADRIC SURFACE

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Abstract: Here we consider integral curves in a smooth quadric surface with either at least one ordinary cusp and general moduli or with several ordinary cusps in general position as only singularities.

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1. Introduction

Here we consider two differently problems concerning cuspidal curves in a smooth quadric surface. In Section 2 we prove the following results.

Theorem 1. Fix integers \(g \geq 5\), \(b \geq a > 0\) and let \(X\) be the general smooth curve of genus \(g\). Let \(Q \in \mathbb{P}^3\) be a smooth quadric surface. There is a morphism \(u : X \to Q\) birational onto its image, with \(u(X)\) of type \((a, b)\) and with \(u(X)\) having at least one ordinary cusp if and only if either \(a = g/2 + 1\) (and hence \(a\) is even) and \(b > 0\) or \(a > g/2 + 1\).

Theorem 2. Fix integers \(g, k, a, b\) such that \(b \geq a > 0\), \(k \geq 2\) and \(2k \geq g\). Let \(X\) be a general \(k\)-gonal curve of genus \(g\) and \(Q \subset \mathbb{P}^3\). There is a morphism \(f : X \to Q\) birational onto its image, with \(f(X)\) of type \((a, b)\) and with \(f(X)\) having an ordinary cusp if and only if \(2b \geq 2g + 2\) and either \(2a \geq g + 1\) or \(a = xk\) for some integer \(x > 0\).
Then we consider cuspidal curves contained in a smooth quadric surface $Q \subset \mathbb{P}^3$ and with cusps supported by general points of $Q$. In Section 3 we will prove the following result.

**Theorem 3.** Fix integers $b \geq a \geq 0$ and set $x := ([a/5])([b/2])$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface and $B \subset Q$ a general subset such that $\sharp(B) = x$. Then there exists an integral curve $Y \subset Q$ of type $(a, b)$ such that $\text{Sing}(Y) = B$ and each singular point of $Y$ is an ordinary cusp with a line of type $(1, 0)$ as a cuspidal curve.

We will briefly consider the case of a quadric cone (see Lemma 5 and Example 2).

We work over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$.

### 2. Proof of Theorem 1 and Theorem 2

**Proof of Theorem 1.** Take an integral curve $T$ of type $(a, b)$ with $X$ as a normalization and let $f : X \to T$ be the normalization map. Hence $f$ is induced by a degree $a$ morphism $f_1 : X \to \mathbb{P}^1$ and a degree $b$ morphism $f_2 : X \to \mathbb{P}^1$. By an elementary form of Brill-Noether theory we must have $2a \geq g + 1$ and if $2b \geq 2a \geq g + 1$ there is such a morphism $f$ which is birational onto its image and with a nodal curve as image. Furthermore, if $2a = 2g + 1$, then $X$ has only finitely many $g_a^1$ and it is easy to check that for general $X$ no two such pencils ramifies at the same point of $X$. In all other cases we easily get $f_1, f_2$ with a common ordinary ramification point, $P$, and such that $f(P') \neq f(P)$ for every $P' \in X \setminus \{P\}$. For such pair map $f$ the curve $f(X)$ has an ordinary cusp at $f(P)$.

**Proof of Theorem 2.** Take an integral curve $T$ of type $(a, b)$ with $X$ as a normalization and let $f : X \to T$ be the normalization map. If $2b \geq 2a \geq g + 1$, then we may repeat the proof of Theorem 1, just quoting [3], Theorem 2.2, instead of “very elementary Brill-Noether theory”. If $2b \leq g + 1$, then there is no such birational morphism by [1], Theorem 2.6. If $2a \leq g$, then we must have $a = xk$ and $2b \geq g + 2$ by [1], Theorem 2.6, and, conversely, for any pair $(xk, b)$ with $2b \geq g + 2$ we have $T$ with an ordinary cusp.

### 3. Cusps in General Position

Let $D \subset A$ be an effective Cartier divisor of $A$ and $Z \subset A$ any closed subscheme of $A$. Let $\text{Res}_D(Z)$ denote the residual subscheme of $Z$ with respect to $D$, i.e. the closed subscheme of $A$ with $\mathcal{I}_{Z,A} : \mathcal{O}_A(-D)$ as its ideal sheaf. By the very definition of residual scheme for any $L \in \text{Pic}(A)$ we have the following exact
sequence:

$$0 \to \mathcal{I}_{\text{Res}_D(Z), A} \otimes L \to \mathcal{I}_{Z, A} \otimes L \to \mathcal{I}_{Z \cap D, D} \otimes L |_D \to 0. \quad (1)$$

From the cohomology exact sequence (1) we get at once the following lemma which is a very elementary version of the so-called Horace Lemma.

**Lemma 1.** Let $A$ be a projective scheme, $D$ an effective Cartier divisor of $A$, $Z$ a closed subscheme of $A$ and $L \in \text{Pic}(A)$. Then:

(i) $h^0(A, \mathcal{I}_{Z, A} \otimes L) \leq h^0(A, \mathcal{I}_{\text{Res}_D(Z), A} \otimes L) + h^0(D, \mathcal{I}_{Z \cap D, D} \otimes L |_D)$;

(ii) $h^1(A, \mathcal{I}_{Z, A} \otimes L) \leq h^1(A, \mathcal{I}_{\text{Res}_D(Z), A} \otimes L) + h^1(D, \mathcal{I}_{Z \cap D, D} \otimes L |_D)$.

**Example 1.** Let $A$ be a smooth quasi-projective surface, $P \in A$ and $D \subset A$ a smooth curve such that $P \in D$. Take local (or formal) coordinates $x, y$ of $S$ around $P$ such that $D = \{ y = 0 \}$ around $P$. Let $W \subset A$ be the zero-dimensional scheme such that $\{ P \} = W_{\text{red}}$ and $W$ has local equation $y^2 = xy^2 = x^3 = 0$ around $P$. Hence $\text{length}(W) = 5$, $\text{length}(W \cap D) = 3$, $\text{Res}_D(W)$ has $y = x^2$ as local equations and hence $\text{length}(\text{Res}_D(W)) = 2$ and $\text{Res}_D(W) \subset D$. In the set-up of [4], Example 2 of §2, $W$ is the associated scheme to an ordinary cusp with $P$ as support and with $D$ as cuspidal tangent. Now assume that $A$ is an open subset of an integral projective surface $S$ and take any $L \in \text{Pic}(S)$. By [4], Lemma 2.4, if $h^1(S, \mathcal{I}_Z \otimes L) = 0$, then a general element of $| \mathcal{I}_Z \otimes L |$ has an ordinary cusp at $P$ with cuspidal tangent contained in $D$, i.e. containing the degree 3 effective divisor of $D$ with $P$ as support.

The following results are a very particular case of [2], Lemma 2.3 (see in particular Figure 1 at p. 308).

**Lemma 2.** Fix $L \in \text{Pic}(Q)$, a line $D \subset Q$ and a closed zero-dimensional subscheme $Z$ of $Q$ and $P \in D$ such that $P \notin Z_{\text{red}}$. Let $W \subset A$ be the zero-dimensional scheme described in Example 1 with $W_{\text{red}} = \{ P \}$, $\text{length}(W \cap D) = 3$, $\text{Res}_D(W) \subset D$ and $\text{length}(\text{Res}_D(W)) = 5$ Set $U := Z \cap W$. Take a general line $R$ in $R$ in the same ruling of $Q$ as $D$ and call $P'$ the point of $R$ such that the line containing $P$ and $P'$ is contained in $Q$. Let $E \subset R$ be the degree two effective divisor of $D$ with $P$ as support. To prove $h^1(Q, \mathcal{I}_U \otimes L) = 0$ (resp. $h^0(Q, \mathcal{I}_U \otimes L) = 0$) it is sufficient to prove $h^1(D, \mathcal{I}_{U \cap D} \otimes (L |_D)) = h^1(Q, \mathcal{I}_{\text{Res}_D(Z) \cup E} \otimes L(-D)) = 0$ (resp. $h^0(D, \mathcal{I}_{U \cap D} \otimes (L |_D)) = h^0(Q, \mathcal{I}_{\text{Res}_D(Z) \cup E} \otimes L(-D)) = 0$).

**Lemma 3.** Let $S$ be a smooth projective surface, $T$ an effective Cartier divisor of $S$ such that $| T |$ has no base points, $P \in T_{\text{reg}}$, $Z \subset S$ a zero-dimensional scheme such that $P \notin Z_{\text{red}}$ and $L \in \text{Pic}(Q)$. Call $mP_T$, $m >
0, the degree $m$ effective divisor of $T$ with $P$ as support. Let $W$ be the associated scheme of an ordinary cusp with as support a general $P' \in S$ and with as cuspidal tangent a general $T' \in |T|$ such that $P' \in T'$. To prove $h^1(S, \mathcal{I}_Z \otimes L) = 0$ (resp. $h^0(S, \mathcal{I}_Z \otimes L) = 0$) it is sufficient to prove $h^1(T, \mathcal{I}_{(Z \cap T)} \otimes (L|_T)) = h^1(S, \mathcal{I}_{\text{Res}_T(Z)} \otimes L(-T)) = 0$ (resp. $h^0(S, \mathcal{I}_{(Z \cap T)} \otimes (L|_T)) = h^0(S, \mathcal{I}_{\text{Res}_T(Z)} \otimes L(-T)) = 0$).

**Lemma 4.** Fix integers $a > 0$, $b > 0$ and set $x := [(a + 1)/5] \cdot [(b + 1)/2]$. Then $h^1(Q, \mathcal{I}_Z(a, b)) = 0$, where $Z$ is the general union of associated schemes for ordinary cusps with as cuspidal tangent a line type $(1, 0)$.

**Proof.** First, assume $a + 1 \equiv 0 \pmod 5$ and $b \equiv 1 \pmod 2$. Fix a line $D$ of type $(1, 0)$ and $(a + 1)/5$ distinct points $P_i \in D$, $1 \leq i \leq (a + 1)/5$. Let $W_i$, $1 \leq i \leq (a + 1)/5$, the associated scheme of an ordinary cusp with $P_i$ as support and $D$ as cuspidal line. Simultaneously, we apply at each $P_i$ Lemma 2 with respect to $W_i$ and itself: to prove $h^1(Q, \mathcal{I}_Z(a, b)) = 0$ we may use instead of the associated schemes of two ordinary cusps with cuspidal tangent of type $(1, 0)$ a virtual scheme $Z_i$ such that $(Z_i)_{\text{red}} = \{P\}$, length$(Z_i) = 10$, length$(Z_i \cap D) = 5$, $\text{Res}_D(Z_i) \subset D$ and length$(\text{Res}_D(Z_i)) = 5$. If $b = 1$, by Lemma 2 we have won. If $b \geq 3$, then by Lemma 2 we reduce to the case $a' := a$, $b' := b - 2$ and $x' := x - (a + 1)/5 = (a' + 1)(b' + 1)/10$. Hence we conclude by induction on $b$. In all other cases we get $h^1(Q, \mathcal{I}_Z(a', b')) = 0$, where $a' := 5[(a + 1)/5]$ and $b' := 2[(b + 1)/2]$ since $a \geq a'$ and $b \geq b'$, we get $h^1(Q, \mathcal{I}_Z(a, b)) = 0$. 

**Proof of Theorem 3.** Let $Z$ be the associated scheme for cusps with the points of $B$ as supporting points and with the corresponding lines of type $(1, 0)$ as their cuspidal tangent. Fix a general $Y \in |\mathcal{I}_Z(a, b)|$. By Lemma 4 (applied to the pair $(a - 1, b - 1)$ we have $h^1(Q, \mathcal{I}_Z(a - 1, b - 1)) = 0$. Since $h^1(Q, \mathcal{I}_Z(a - 1, b - 1)) = 0$, we have dim$(|\mathcal{I}_Z(a, b)|) = (a + 1(b + 1) - 5x - 1$ and $Y$ has an ordinary cusp at each point of $B$ with the corresponding line of type $(1, 0)$ as its cuspidal tangent (use Example 1). Since $h^1(Q, \mathcal{I}_Z(a - 1, b - 1)) = 0$, we easily see that $|\mathcal{I}_Z(a, b)|$ has no base points outside $B$. Hence $Y$ is smooth outside $B$ by Bertini Theorem. Since $Y$ is unibranch at each point of $B$, a very easy dimensional count gives the irreducibility of $Y$.

From now on we will consider the case of a quadric cone.

**Lemma 5.** Let $S$ denote the Hirzebruch surface $F_2$. We recall that $\text{Pic}(S)$ is freely generated as an additive group by the class, $f$, of the ruling of $S$ and the class of the section, $h$, of the ruling with negative self-intersection. Hence $h^2 = -2$, $h \cdot f = 1$ and $f^2 = 0$. Fix integers $a \geq 2$ and $b \geq 2a$. Let $A \subset S$ (resp. $B \subset S$) be the union of $[a^2/5]$ (resp. $[(a - 1)^2/5]$) general points of $S$. There
are curves $C, D \in |ah + bf|$ such that $C$ has an ordinary cusp at each point of $A$, $D$ has an ordinary cusp at each point of $B$, $D$ is integral and $\text{Sing}(D) = B$.

**Proof.** First, we will show that for the general union $Z$ of $[a^2/5]$ associated schemes of ordinary cusps we have $h^1(S, \mathcal{I}_Z(ah + bf)) = 0$. Since the cases $a = 2, 3$ are easy by the rounding down, we will always assume $a \geq 4$. It is sufficient to prove $h^1(S, \mathcal{I}_Z(ah + 2af)) = 0$. Since 3 and 2 are coprime, there are uniquely determined integers $\alpha, \beta$ such that $3\alpha + 2\beta = 2a + 1$ and $0 \leq \beta \leq 2$. Hence $\beta = 0$ (and hence $\alpha = (2a + 1)/3$) if and only if $a \equiv 1 \pmod{3}$, $\beta = 1$ (and hence $\alpha = (2a - 1)/3$) if and only if $a \equiv 2 \pmod{3}$ and $\beta = 2$ (and hence $\alpha = a - 1$) if and only if $a \equiv 0 \pmod{3}$. Notice that $\alpha \geq 0$. Let $T$ be any irreducible element of $[h + 2f]$. Notice that $T \cong \mathbb{P}^1$ and $\mathcal{O}_T(ah + 2af)$ has degree $2a$. Fix $\alpha + \beta$ distinct points $P_i, Q_j \in T$, $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$. Insert an associated scheme for ordinary cusp with $T$ as a cuspidal tangent at each $P_i$ and apply Lemma 3 (the so-called $(2, 3)$-trick) for the same associated schemes at each point $Q_j$. We need to control for $\mathcal{O}_S((a - 1)h + 2(a - 1)f)$ a zero-dimensional scheme $E \subset T$ of length $2\alpha + 3\beta$ with $\alpha$ connected components with length 2 supported by each point $P_1, \ldots, P_\alpha$ and $\beta$ connected components with length 3 supported by each $Q_1, \ldots, Q_\beta$. Here we use $3\beta + 2\alpha \leq 2a - 1$. Then we continue in the same way with integers $\alpha' + \beta'$ such that $3\alpha' + 2\beta' = 2a - 1 - 2\alpha - 3\beta$ and $0 \leq \beta' \leq 2$. Let $W$ be the general union of $[(a - 1)^2/5]$ associated schemes of ordinary cusps. We just saw that $h^1(S, \mathcal{I}_W((a - 1)h + 2(a - 1)f)) = 0$. Let $D$ be a general element of $[\mathcal{I}_W(ah + bf)]$. We know that $D$ has an ordinary cusp at each point of $B$. Since $\mathcal{O}_S(h + (b - 2a + 2)f)$ is very ample, it is easy to check that $B$ is the base locus of $[\mathcal{I}_W(ah + bf)]$. Hence by Bertini Theorem $D$ is smooth outside $B$. Since $B = \text{Sing}(D)$ and any cusp is unibranch, $D$ is irreducible. \qed

**Example 2.** Here we consider the case of an integral quadric cone $Q \subset \mathbb{P}^3$ and call $P$ its vertex. We recall that $\text{Pic}(Q) = \mathbb{Z}[\mathcal{O}_Q(1)]$, but that there are Weil divisors $C \in Q$ which are not Cartier divisor. Let $u : S \to Q$ be the blowing-up of $Q$ at $P$. Set $h := u^{-1}(P)$. We have $S \cong F_2$ and $\text{Pic}(Q)$ is freely generated by the class of $h$ and the class of the ruling, $f$, of $S$. Let $C \cong Q$ be an integral curve and $C' \subset S$ its strict transform in $S$. Call $m \geq 0$ the multiplicity of $C$ at $P$. If $C$ is a line, then $C' \in |f|$. If $C$ is not a line, then any line through $P$ intersects $C$ outside $P$ in $a > 0$ points (counting multiplicities) and $C' \in |ah + (2a + m)f|$. $P \not\in C$ if and only if $m = 0$. $P \in C$ and $C$ is smooth at $P$, then $m = 1$. If $C$ has a double point at $C$, then $m = 2$. Hence if $C$ has only ordinary cusps as singularities, then $m \leq 2$. By Lemma 5 if $m = 0, 1$ for a general $B \subset Q$ with $\sharp(B) = [(a - 1)^2/5]$ there is an integral curve $Y \subset Q$ whose
strict transform is in $|ah + (2a + m)f|$ and with an ordinary cusp at each point of $B$ as only singularities. If $m = 2$ we may easily check that we may find $D$ as above, but with an ordinary cusp at the vertex $P$.

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References


