HERMITE AND BIRKHOFF “ALMOST UNIFORM” BIVARIATE INTERPOLATION

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Abstract: Here we consider an almost regular bivariate Birkhoff interpolation problem for polynomials which is uniform for almost all nodes.

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1. Bivariate Interpolation

We fix positive integers $n, d$ and a field $K$. We will always take derivatives in $n$ variables such that the order in each variable is at most $d$. If either $\text{char}(K) = 0$ (i.e. $K$ contains $\mathbb{Q}$) or $\text{char}(K) > d$ these derivatives will be the usual derivatives, while if we take a derivative of order $\geq \text{char}(K)$ instead of the usual derivatives we will use the so-called Hasse derivatives (see [2], §3), which are the derivations needed in low characteristic to get the usual Taylor expansion of a polynomial of degree at most $d$ at any $t \in K$. Except for the end of the paper we consider only bivariate polynomials. If $K$ is finite we will often need to assume $\sharp(K)$ not too small (say $\sharp(K) \geq d$) to have enough nodes with coefficients in $K$ to do the interpolation game. In [5] there are described in details the following multivariate Hermite interpolation problem. Take all polynomials in $n$ variables $z_1, \ldots, z_n$ with total order at most $d$. At each node $P_j$ take all partial derivatives of total order up to a prescribed integer $m_j$ (Hermite interpolation of type total degree). As stressed in [5], p. 31, it
would be easier to consider Hermite interpolation problems of uniform type, i.e. for which the prescribed maximal order of the derivatives is the same at all nodes. However, as stressed in [5], except the case of Lagrangian interpolation, this condition may be satisfied for a very small class of integers \( d \), where \( d \) is the prescribed upper bound for the degree of the polynomials. We think that it would be nice at least to prescribe the same order of derivatives at almost all nodes and we obtain this condition in the set-up of Birkhoff interpolation (see Theorem 1 and Theorem 2). In [5] R. Lorentz describe the deep difference between regularity and almost regularity for an interpolation problem. Here we only consider almost regularity, but in the statement of Theorem 1 we also give some hint on how to choose the winning nodes for the interpolation problems. Obviously, the same set-up works for Birkhoff interpolation IF (as we will do in this paper) fix the coordinate system \( x, y \). Fix \( P \in K^2 \) and integers \( ab \). A flat tower of type \( (a, b) \) (or an \( (a, b) \)-tower) with \( P \) as its support (with respect to the fix coordinate system \( (x, y) \)) is the interpolation datum on a vector space of polynomials obtained evaluating at \( P \) all partial derivatives of order at most \( a \) in which the partial derivative with respect to \( y \) occurs at least \( b \) times. Hence an \( (a, b) \)-tower gives \( a(a+1)/2-b(b+1)/2 \) linear conditions to the interpolation game.

**Theorem 1.** Fix integers \( d \geq 3m > 0 \). Set \( k_i := \lfloor(d+1-im-m)/(m+1)\rfloor \) and \( e_i := d+1-im-m-k_i(m+1), 1 \leq i \leq s := \lfloor(d+1-m)/m\rfloor \). Then the Birkhoff interpolation problem for polynomials in two variables with total degree at most \( d \) and with respect to derivatives with total order at most \( m \) at \( 2(\sum_{i=1}^{s} k_i) \) points of \( K^2 \), another one with total order at most \( a := d+1-sm-m \leq m \) and with \( s \) partial towers of type \((m+1+e_i,e_i), 1 \leq i \leq s \) is almost regular. Furthermore, we may find a solution of this interpolation problem in the following way, in which only the half of the \( y \)-coordinates of the points must be “generic”, while the \( x \)-coordinates of all points are prescribed in advance and are arbitrary. Fix \( x_1, \ldots, x_{k_1+1} \in K \) such that \( x_i \neq x_j \) for all \( i \neq j \). Then fix \( s \) distinct \( y_{2t-1} \in K, 1 \leq t \leq s \). Then (but only then) choose a general \( s \)-ple of points of \( K \), say \( y_{2t}, 1 \leq t \leq s \). Set \( P_{i,j} := (x_i, y_j) \in K^2 \). Describe all the partial derivatives at that point up to order \( m+1 \) at the points \( P_{i,j} \) such that \( 1 \leq i \leq 2s \) and \( 1 \leq j \leq k_j \). Take a partial tower of type \((m+1+e_i,e_i)\) at the point \( P_{k_{j+1},2j-1} \). Choose \( Q = (u,v) \in K^2 \) with \( v \neq y_i \) for all \( i \), and use \( Q \) to evaluate the partial derivatives up to order \( a-1 \).

**Remark 1.** In the statement of Theorem 1 one can take \( a(a+1)/2 \) general \( Q_i \in K^2, 1 \leq i \leq a(a+1)/2 \), and use the Lagrange interpolation problem at these points, instead of the Hermite one with derivatives up to order \( a-1 \) at
Remark 2. Let $D$ be an affine line of the plane $K^2$, $d$ a positive integer and $Z \subset K^2$ a configuration (points with multiplicities $m_j$, i.e. conditions on the derivatives at the prescribed points up to the prescribed order $m_j - 1$, or, more generally, partial towers of type $(Q_j, a_j, b_j)$ supported), say $\{(Q_j, m_j)\}$. The residual scheme (or residual configuration) Res$_D$ is the configuration $\{(Q_j, a'_j, b'_j)\}$ described in the following way: set $a'_j := a_j$ and $b'_j := b_j Q_j \notin D$ and $a'_j := a_j - 1, b'_j := b_j$ if $Q_j \in D$, with the convention that we ignore $Q_j$ in Res$_D$ if $a'_j = b'_j$, i.e. if $a_j = b_j + 1$. Call $\tau$ the sum of the integers $a_j$'s with $Q_j \in D$, $\alpha$ (resp. $\beta$) the number of conditions given by $Z$ (resp. Res$_D$) to polynomials in two variables of degree at most $d$ (resp. $d - 1$). The following statements are elementary forms of the so-called Horace Lemma, but in this case they may be checked directly. If $\tau \leq d + 1$, then $\alpha \geq \beta + d + 1 - \tau$. If $\tau \geq d + 1$, then $\alpha = \beta + d + 1$. In particular, if $\tau = d + 1$, then $Z$ uniquely solves bivariate interpolation in degree up to $d$ if and only if Res$_D$ uniquely solves bivariate interpolation up to degree $d - 1$.

Remark 3. Fix integers $m > 0$, $k_1 > 0$, $e_1 \geq 0$ and set $d := k(m + 1) + m + 1 + e_1 - 1$. Fix $y_1 \in K$ and $x_1, \ldots, x_{k_1 + 1} \in K$ with $x_i \neq x_j$ for all $i \neq j$. Then take a “general” $y_2 \in K$. Set $A_j := (x_j, y_1), 1 \leq j \leq k_1 + 1$, and $B_j := (x_j, y_2), 1 \leq j \leq k_1$. Call $Z$ the interpolation datum $\bigcup_{i=1}^{k_1}(A_i, m) \cup \bigcup_{j=1}^{k_1}(B_j, 1) \cup (A_{k_1+1}, m+1+e_1, e_1)$. For general $y_2$ the interpolation datum gives $k_1 m(m + 1) + (m + 2 + e_1)(m + 1 + e_1)/2 - e_1(e_1 + 1)/2$ independent conditions (i.e. the maximal possible number of independent conditions) for bivariate polynomials of degree at most one for the following reason. For $1 \leq j \leq k_1$ make $B_j$ collide to $A_j$ along the line $\{x = x_j\}$ and do that simultaneously at all $k_1$ points $B_j$. We apply [1], Lemma 2.3, (see in particular Figure 1 at p. 308 of [1]) first with the integer 1, then with the integer 12, and so on increasing by one at each step. Since we do the $k_1$ collision simultaneously using parallel lines, a general $y_2$ do the job for all $k_1$ points $B_j, 1 \leq j \leq k_1$.

We stress that (according to their authors) the key [1], Lemma 2.3, was inspired by [3] and [4].

Proof of Theorem 1. Use the configuration, $Z$, described in the statement of the theorem. Make the collision of each line $\{y = y_{2j}\}, 1 \leq j \leq s$, with the line $\{y = y_{2j-1}\}$ preserving the $x$-coordinate. At each $P_{1,2j-1}$ appearing with multiplicity $m$ we add one condition, after one step (i.e. in the residual scheme) two conditions, and so on, slicing a point with multiplicity $m$. In this way from two points with multiplicity $m$ we obtain $m$ times a virtual scheme whose intersection with the line $\{y = y_{2j-1}\}$ has multiplicity $m + 1$. Call $Z'$
this virtual set of interpolation conditions. Notice that the restriction of \( Z' \) to the line \( \{ y = y_{2j-1} \} \) is an interpolation datum in one variable with \( k_j \) points with multiplicity \( m + 1 \) and one stratum of a \( (m + 1 + e_j, e_j) \) tower; in this way at each of the \( m \) steps inductive steps related to the line \( \{ y = y_{2j-1} \} \) we have exactly the right number of conditions to solve uniquely the corresponding one-variable interpolation problem.

We conclude this note with the remark that, with obvious modifications, [5], Theorem 3.4.2, is true over any field, i.e. that the following statement is true.

**Theorem 2.** Let \((E, P_S)\) be an interpolation scheme over a field \( K \) and call \( a \) the maximal order of its derivatives. Assume either \( \text{char}(K) = 0 \) or \( \text{char}(K) > a \). Then \((E, P_S)\) is regular if and only if \( E \) is an Abel matrix with respect to \( S \); for the “only if” part assume \( K \) infinite.

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**References**


