International Journal of Pure and Applied Mathematics
Volume 18 No. 1 2005, 49-57

APPLICATION OF ADOMIAN METHOD ON THE ELASTIC WAVE PROPAGATION IN A SEMI-INFINITE ELASTIC ROD OF RANDOMLY VARYING CROSS SECTION

S. Saha Ray[1 §], R.K. Bera[2]

[1]B.P. Poddar Institute of Management and Technology
137, V.I.P. Road, Poddar Vihar, Kolkata, 700052, INDIA
e-mail: santanusaharay@yahoo.com

[2] Heritage Institute of Technology
Chowbaga Road, Anandapur, Kolkata, 700107, INDIA
e-mail: rasajit@hotmail.com

Abstract: The present analysis seeks to investigate the mean and variance of the displacement distribution in a semi-infinite homogeneous elastic rod of randomly varying cross-section due to a time-dependent displacement input at one end. Laplace transform technique is applied to remove the time variable. Then the solution of this wave problem is obtained in the transformed plane by Adomian decomposition method [1] in the form of the series for small random variations in the cross-sectional area along the rod. Three particular cases of different continuous probability distributions of the random variable are studied and the effect of randomness is observed.

AMS Subject Classification: 73D70, 73D35
Key Words: wave propagation, random, variable cross section, adomian method

1. Introduction

Elastic wave propagation in semi-infinite elastic rods having randomly varying

Received: September 1, 2004

§Correspondence author

shapes has received considerable attention in recent years [2]. In the present discussion the method of Adomian has been used for the solution of the problem. It may be mentioned in this connection that the elastic wave propagation in non-homogeneous rods has been studied by many researchers [3], [4], [5]. But the present problem has not been solved by any one of them by Adomian method. Keller [6], [7] has studied several problems on wave motion in random media in geometrical optics, where the small random variations in homogeneity of the medium has been considered. Instead of considering the randomness in elastic properties, the randomness of the shape of the cross-sectional area along the rod has been considered here. In this paper, the so-called "honest" method has been utilized. In this method the solution for the displacement $u(x, \alpha, t)$, where x is the space co-ordinate, t is the time and α is a parameter ranging over the probability space S with probability density $p(\alpha)$, is first determined for each value of α and then its mean value and other statistics such as variance, etc. are computed. The solution is expressed in the form of series by Adomian decomposition method to see the efficiency and effectiveness of the method. There is a slight random variation in the cross-sectional area of the rod while the elastic property and boundary conditions are assumed to be deterministic in nature. Three specific cases of continuous probability distributions such as gamma distribution, uniform and linear type distributions are studied.

2. Formulation of the Problem: Governing Equations

We consider a semi-infinite elastic rod, out of the large sample under consideration, occupying the region $x \geq 0$. The end $x = 0$ of the rod is subjected to a time-dependent displacement input. The longitudinal displacement $u(x, t)$ of such a rod with varying cross-section area $A(x)$ is obtained as a solution of the problem:

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = \rho A\frac{\partial^2 u}{\partial t^2}\,, \tag{1}$$

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0\,, \tag{2}$$

$$u(0,t) = f_1(t)\,, \tag{3}$$

$$\text{and } u(x,t) \to 0 \quad \text{as} \quad x \to \infty\,. \tag{4}$$

Here E, the Young's modulus and ρ the density are supposed constants and are deterministic in nature. The cross-sectional area "A" is taken as a function of x and a parameter α. This parameter α ranges over a probability space S with certain probability density $p(\alpha)$. We assume randomness in the cross-section in the form

$$A \equiv A(x,\alpha) = A_0 + \epsilon g(x,\alpha)\,, \tag{5}$$

where $g(x,\alpha)$ is a bounded function of x and α. Since the cross-sectional area is of a random nature with only slight variations, the parameter ϵ is supposed to be very small. The equation (1) with (5), then reduces to

$$(1+\frac{\epsilon}{A_0}g(x,\alpha))\frac{\partial^2 u}{\partial x^2} + \frac{\epsilon}{A_0}\frac{dg}{dx}\frac{\partial u}{\partial x} = \frac{1}{c^2}(1+\frac{\epsilon}{A_0}g(x,\alpha))\frac{\partial^2 u}{\partial t^2}\,, \tag{6}$$

where $c = \sqrt{E/\rho}$ is the longitudinal elastic wave velocity in the rod. Taking Laplace transform of equation (6) and using equation (2), we obtain

$$(1+\frac{\epsilon}{A_0}g(x,\alpha))\frac{d^2\overline{u}}{dx^2} + \frac{\epsilon}{A_0}\frac{dg}{dx}\frac{d\overline{u}}{dx} = \frac{s^2}{c^2}(1+\frac{\epsilon}{A_0}g(x,\alpha))\overline{u}\,. \tag{7}$$

This implies that

$$\frac{d^2\overline{u}}{dx^2} + \frac{\epsilon}{A_0}\frac{dg}{dx}\frac{d\overline{u}}{dx} = \frac{s^2}{c^2}\overline{u}\,, \tag{8}$$

where ϵ/A_0 is assumed to be very small.

The boundary conditions equation (3) and equation (4) now reduces to

$$\overline{u}(0,s) = \overline{f_1}(s)\,, \tag{9}$$

$$\overline{u}(x,s) \to 0 \quad \text{as} \quad x \to \infty. \tag{10}$$

3. Approximate Solution by Adomian Method

To solve the equation (8) by Adomian method, we rewrite the equation (8) in the form

$$L\overline{u} + R\overline{u} = 0\,, \tag{11}$$

where $L \equiv (\frac{d^2}{dx^2} - \frac{s^2}{c^2})$ is an invertible linear operator and

$$R = \frac{\epsilon}{A_0}\frac{dg}{dx}\frac{d}{dx}\,. \tag{12}$$

From (11) we can write

$$L\overline{u} = -R\overline{u}, \quad \text{so that } \overline{\phi} = \psi - L^{-1}(R\overline{u})\,, \tag{13}$$

where $L\psi = 0$. This gives

$$\psi = c_1 e^{sx/c} + c_2 e^{-sx/c}\,, \tag{14}$$

where c_1 and c_2 are arbitrary constants.

In light of the Adomian decomposition method we assume that

$$\overline{u} = \sum_{n=0}^{\infty} \overline{u}_n \quad \text{with} \quad \overline{u}_0 = c_1 e^{sx/c} + c_2 e^{-sx/c}\,. \tag{15}$$

Hence we can write

$$\overline{u}_{n+1} = -L^{-1}(R\overline{u}_n) \quad (n \geq 0)\,, \tag{16}$$

$$\text{with} \quad \overline{u}_0 = c_1 e^{sx/c} + c_2 e^{-sx/c}\,. \tag{17}$$

Suppose, a time-dependent displacement input $f_1(t)$ is applied at $x = 0$, where

$$f_1(t) = U_0(1 - cos\frac{\pi t}{\tau}) \quad \text{for} \quad 0 \leq t \leq \tau\,, \tag{18}$$

$$= 2U_0 \quad \text{for} \quad t \geq \tau\,. \tag{19}$$

Then

$$\overline{f_1}(s) = \frac{\pi^2 U_0}{\tau^2 s(s^2 + \pi^2/\tau^2)}(1 + e^{-s\tau})\,. \tag{20}$$

We assume randomness in the cross-sectional area $A(x,t)$ in the from $A = A_0 + \epsilon g(x,\alpha)$, where

$$g(x,\alpha) = e^{-\beta x} f(\alpha)\,, \tag{21}$$

where β is a positive constant and α is a random variable.

From equation (16), we obtain

$$\overline{u}_1 = -L^{-1}(R\overline{u}_0) = -(\frac{\epsilon}{A_0})\frac{s}{c}f(\alpha)[c_1\frac{e^{(s/c-\beta)x}}{(\frac{2s}{c}-\beta)} + c_2\frac{e^{-(s/c+\beta)x}}{(\frac{2s}{c}+\beta)}]\,. \tag{22}$$

Therefore from equation (15), we can write

$$\overline{u} = \overline{u}_0 + \overline{u}_1 = c_1 e^{sx/c} + c_2 e^{-sx/c} - (\frac{\epsilon}{A_0})\frac{s}{c} f(\alpha)[c_1 \frac{e^{(s/c-\beta)x}}{(\frac{2s}{c}-\beta)} + c_2 \frac{e^{-(s/c+\beta)x}}{(\frac{2s}{c}+\beta)}] \quad (23)$$

upto the first order of ϵ/A_0.

After satisfying the boundary and regularity conditions equation (9) and equation (10), we obtain

$$c_1 = 0 \quad (24)$$

and

$$c_2 = [1 - \frac{(\frac{\epsilon}{A_0})\frac{s}{c} f(\alpha)}{(\frac{2s}{c}+\beta)}]^{-1} \overline{f}_1(s) = [1 + \frac{(\frac{\epsilon}{A_0})\frac{s}{c} f(\alpha)}{(\frac{2s}{c}+\beta)}]\overline{f}_1(s) \quad (25)$$

upto the first order of ϵ/A_0. Therefore equation (23) becomes

$$\overline{u} = [1 + \frac{(\frac{\epsilon}{A_0})\frac{s}{c} f(\alpha)}{(\frac{2s}{c}+\beta)}]\overline{f}_1(s) e^{-sx/c} - (\frac{\epsilon}{A_0})\frac{s}{c} f(\alpha)[1 + \frac{(\frac{\epsilon}{A_0})\frac{s}{c} f(\alpha)}{(\frac{2s}{c}+\beta)}] \times \overline{f}_1(s) \frac{e^{-(s/c+\beta)x}}{(\frac{2s}{c}+\beta)} = \overline{f}_1(s) e^{-sx/c} + (\frac{\epsilon}{A_0})\frac{\frac{s}{c} f(\alpha)\overline{f}_1(s) e^{-sx/c}}{(\frac{2s}{c}+\beta)}(1 - e^{-\beta x})\,. \quad (26)$$

Equation (20) with equation (26) becomes

$$\overline{u} = \frac{\pi^2 U_0 e^{-sx/c}}{\tau^2 s(s^2 + \pi^2/\tau^2)}(1 + e^{-s\tau}) + (\frac{\epsilon}{A_0})\frac{\pi^2 U_0 f(\alpha)(1 - e^{-\beta x})}{2\tau^2}[\frac{e^{-sx/c} + e^{-s(\tau + x/c)}}{(s + \beta c/2)(s^2 + \pi^2/\tau^2)}]\,. \quad (27)$$

Taking inverse transform of equation (27), we obtain

$$u(x, \alpha, t) = u_0(x, t) + (\epsilon/A_0) u_1(x, t) f(\alpha)\,, \quad (28)$$

where

$$u_0(x, t) = U_0[H(t - x/c) - \cos(\frac{\pi}{\tau}(t - x/c))H(t - x/c) + H(t - \tau - x/c) - \cos(\frac{\pi}{\tau}(t - \tau - x/c))H(t - \tau - x/c)]\,, \quad (29)$$

and

$$u_1(x,t) = \frac{\pi^2 U_0(1-e^{-\beta x})}{2\tau^2}\Big[\frac{e^{\frac{-\beta c}{2}(t-x/c)}}{(\frac{\beta^2c^2}{4}+\frac{\pi^2}{\tau^2})}H(t-x/c)$$
$$-\frac{\cos(\frac{\pi}{\tau}(t-x/c))}{(\frac{\beta^2c^2}{4}+\frac{\pi^2}{\tau^2})}H(t-x/c)+\frac{(\frac{\beta c}{2})\sin(\frac{\pi}{\tau}(t-x/c))}{(\frac{\beta^2c^2}{4}+\frac{\pi^2}{\tau^2})\frac{\pi}{\tau}}H(t-x/c)$$
$$+\frac{e^{\frac{-\beta c}{2}(t-\tau-x/c)}}{(\frac{\beta^2c^2}{4}+\frac{\pi^2}{\tau^2})}H(t-\tau-x/c)-\frac{\cos(\frac{\pi}{\tau}(t-\tau-x/c))}{(\frac{\beta^2c^2}{4}+\frac{\pi^2}{\tau^2})}H(t-\tau-x/c)\Big]. \quad (30)$$

4. Computation of Mean of the Displacement Distribution

Let us assume that the random variable α ranges over the probability space S with probability density $p(\alpha)$. Then we can determine the mean displacement distribution in the system of rods as

$$< u >=< u_0 > +(\epsilon/A_0) < u_1 >< f(\alpha) >, \quad (31)$$

where the mean of $f(\alpha)$ is defined by

$$< f >= \int_S f(\alpha)p(\alpha)d\alpha\,. \quad (32)$$

4.1. Three Different Distributions

(a) Let us take the random non-homogeneous function $f(\alpha)$ in the form

$$f(\alpha) = k\alpha\,, \quad (33)$$

where k is a positive constant and the distribution of the random variable α is given by the probability density function

$$p(\alpha) = \begin{cases} \frac{1}{2}(1+\alpha), & -1<\alpha<1\,, \\ 0, & \text{elsewhere}\,. \end{cases} \quad (34)$$

Let

$$< f >= \int_{-1}^{1} f(\alpha)p(\alpha)d\alpha = \frac{k}{3}\,. \quad (35)$$

From equation (31) and equation (35), we obtain

$$< u >= u_0(x,t) + (\epsilon/A_0)u_1(x,t)\frac{k}{3} \tag{36}$$

upto the first order of (ϵ/A_0). In this case we conclude from equation (36) that the effect of randomness defined by equation (33) and equation (34) on the wave motion is to multiply the non-homogeneous part of the mean displacement distribution by the constant factor $k/3$.

(b) Let us take the random non-homogeneous function in the form

$$f(\alpha) = e^{-k\alpha}\,, \tag{37}$$

where k is a positive constant and the distribution of the random variable α is given by the probability density function

$$p(\alpha) = \begin{cases} \frac{\alpha^{m-1}e^{-\alpha/n}}{\Gamma(m)n^m}, & 0 < \alpha < \infty\,, \\ 0, & \text{elsewhere}\,, \end{cases} \tag{38}$$

where m and n are positive constants. Let

$$< f >= \int_0^\infty f(\alpha)p(\alpha)d\alpha = (1+kn)^{-m}\,. \tag{39}$$

From equation (31) and equation (39), we obtain

$$< u >= u_0(x,t) + (\epsilon/A_0)u_1(x,t)(1+kn)^{-km} \tag{40}$$

upto the first order of (ϵ/A_0). In this case we conclude from equation (40) that the effect of the continuous random variable defined by equation (37) and equation (38) on the propagation of wave in the system of rods is to multiply the non-homogeneous part of the mean displacement distribution by the constant factor $(1+kn)^{-m}$.

5. Computation of the Variance of the Displacement Distribution

From the equation (28), we see that the approximate solution for the displacement upto the first order of (ϵ/A_0) is

$$u(x,\alpha,t) = u_0(x,t) + (\epsilon/A_0)u_1(x,t)f(\alpha)\,. \tag{41}$$

This determined displacement is a random function for each α with probability density $p(\alpha)$. By definition

$$Var(u) = E[(u - < u >)^2]\,, \tag{42}$$

where E stands for the expectation of the random variate concerned.

$$\begin{aligned} \mathrm{Var}\,(u) &= (\epsilon/A_0)^2 {u_1}^2(x,t) E[(f - < f >)^2] \\ &= (\epsilon/A_0)^2 {u_1}^2(x,t)[E(f^2) - \{< f >\}^2]\,. \end{aligned} \tag{43}$$

Introducing equations (33), (34) and (35) in equation (43), we obtain

$$\mathrm{Var}\,(u) = (\epsilon/A_0)^2 {u_1}^2(x,t) 2k^2/9\,. \tag{44}$$

Again from equations (37), (38) and (39), we obtain

$$\mathrm{Var}\,(u) = (\epsilon/A_0)^2 {u_1}^2(x,t)[(1+2kn)^{-m} - (1+2kn)^{-2m}]\,. \tag{45}$$

6. Concluding Remarks

The solution for the displacement $u(x,\alpha,t)$ by the decomposition method of Adomian is very efficient, simple and new and also very easy to manipulate. The iteration techniques used elsewhere for the solution of similar problems is not simpler than this method. Furthermore, it may be mentioned in this connection that the convergence is also very fast and the results obtained here can be easily identified with those obtained elsewhere.

References

[1] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston (1994).

[2] S.K. Roychaudhri, G.D. Sain, A note on the elastic wave propagation in a semi-infinite elastic rod of randomly varying cross-section, *Indian J. Theoretical Physics*, **32**, No. 3 (1984), 219-228.

[3] R.Y. Vasudeva, R.K. Bhaskara, Propagation of a pulse in a non-homogeneous rod with varying cross-section, *J. Appl. Mech., ASME*, **45** (1978), 942-944.

[4] U.S. Lindholm, K.D. Doshi, Wave propagation in an elastic non-homogeneous bar of finite length, *J. Appl. Mech., ASME*, **32** (1965), 135-142.

[5] G. Singh, A. Singh, Wave propagation in a linear random non-homogeneous viscoelastic semi-infinite rod, *Indian J. of pure and Applied Math.*, **11** (1980), 1095-1104.

[6] J.B. Keller, Wave propagation in random media, In: *Proc. of Sym. in Appl. Math.*, **13**, Am. Math. Soc., New York (1960), 227-246.

[7] J.B. Keller, Stochastic equations and wave propagation in random media, In: *Proc. of Sym. in Appl. Math.*, **16**, Am. Math. Soc., New York (1964), 145-170.