International Journal of Pure and Applied Mathematics

Volume 18 No. 1 2005, 59-70

THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATION METHODS WITH ERRORS FOR LIPSCHITZIAN ϕ-STRONGLY ACCRETIVE OPERATORS

Zeqing Liu[1], Li Wang[2], Soo Hak Shim[3], Shin Min Kang[4 §]

[1]Department of Mathematics
Liaoning Normal University
P.O. Box 200, Dalian, Liaoning, 116029, P.R. CHINA
e-mail: zeqingliu@dl.cn

[2]Department of Science
Shenyang Institute of Aeronautical Engineering
P.O. Box 148, Shenyang, Liaoning, 110034, P.R. CHINA

[3,4]Department of Mathematics
Research Institute of Natural Science
Gyeongsang National University
Chinju, 660-701, KOREA
[4]e-mail: smkang@nongae.gsnu.ac.kr

Abstract: In this paper, we establish the equivalence between the convergences of Mann and Ishikawa Iteration methods with errors for Lipschitzian ϕ-strongly accretive and ϕ-strongly pseudocontractive operators, respectively, in Banach spaces. An example is also included to dwell upon the importance of the results obtained.

AMS Subject Classification: 47H10, 47H15

Key Words: Lipschitzian ϕ-strongly accretive operator, ϕ-Lipschitzian strongly pseudocontractive operator, Mann iteration sequence with errors, Ishikawa iteration sequence with errors

Received: September 12, 2004

[§]Correspondence author

1. Introduction and Preliminaries

Let X be a real Banach space and J denote the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\}, \quad \forall x \in X,$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let I be the identity operator on X. An operator T with domain $D(T)$ and range $R(T)$ in X is called *strongly accretive* if there exists a constant $k > 0$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2.$$

Without loss of generality, we may assume that $k \in (0,1)$. The operator T is called *accretive* if $k = 0$ in above inequality. Furthermore, the operator T is called *ϕ-strongly accretive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|. \tag{1.1}$$

An operator T is said to be *pseudocontractive* (resp., *strongly pseudocontractive*, *ϕ-strongly pseudocontractive*) if $I - T$ is accretive (resp., strongly accretive, ϕ-strongly accretive).

In [1]-[17] the authors studied the existence, uniqueness and iterative approximations of solutions and fixed points for nonlinear equations $Tx = f$ and nonlinear operators T, respectively, under various conditions. Osilike [11] proved that the class of strongly accretive operators is a proper subclass of the class of ϕ-strongly accretive operators. Xu [16] introduced the following Ishikawa iteration sequence with errors:

$$\begin{aligned} &x_0 \in X, \quad y_n = a'_n x_n + b'_n S x_n + c'_n \delta_n, \\ &x_{n+1} = a_n x_n + b_n S y_n + c_n \sigma_n, \quad n \geq 0, \end{aligned} \tag{1.2}$$

where $S : X \to X$ is an operator, $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in X and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$. In case $a'_n = 1$ and $b'_n = c'_n = 0$ for $n \geq 0$, then (1.2) reduces to the following Mann iteration sequence with errors

$$u_0 \in X, \quad u_{n+1} = a_n u_n + b_n S u_n + c_n \sigma_n, \quad n \geq 0. \tag{1.3}$$

Liu-Xu-Cho [3] investigated that, when $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ satisfy the following conditions

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad n \geq 0, \tag{1.4}$$

$$\sum_{n=0}^{\infty} c_n < \infty, \ \sum_{n=0}^{\infty} b_n b'_n < \infty, \ \sum_{n=0}^{\infty} b_n c'_n < \infty, \ \sum_{n=0}^{\infty} b_n^2 < \infty, \tag{1.5}$$

$$\sum_{n=0}^{\infty} b_n = \infty, \tag{1.6}$$

and $T : X \to X$ is Lipschitzian ϕ-strongly accretive, then the Ishikawa iteration sequence with errors defined by (1.2) converges strongly to the unique solution of the equation $Tx = f$, for any given $f \in X$, where $S : X \to X$ is defined by $Sx = f + x - Tx$ for all $x \in X$. If $c_n = c'_n = 0$, $b_n = \alpha_n$, $a_n = 1 - \alpha_n$, $b'_n = \beta_n$ and $a'_n = 1 - \beta_n$, then Ishikawa and Mann iteration sequences with errors reduce to the usual Ishikawa and Mann iteration sequences as following:

$$\begin{aligned} &x_0 \in X, \quad y_n = (1 - \beta_n)x_n + \beta_n S x_n, \\ &x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n, \quad n \geq 0, \end{aligned} \tag{1.7}$$

and

$$u_0 \in X, \quad u_{n+1} = (1 - \alpha_n)u_n + \alpha_n S u_n, \quad n \geq 0, \tag{1.8}$$

where $S : X \to X$ is an operator and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$. Recently, Rhoades-Soltuz [12] proved that certain Mann and Ishikawa iteration schemes are equivalent for Lipschitzian strongly pseudocontractive operators. Rhoades-Soltuz [13] showed that the convergence of Mann iteration is equivalent to the convergence of Ishikawa iteration for some classes of non-Lipschitzian operators. Rhoades-Soltuz [14] discussed the equivalence between the convergences of Ishikawa and Mann iterations for asymptotically pseudocontractive operators. Chang-Cho-Kim [1] established a few equivalence conditions between the convergence of modified Picard, modified Mann, and modified Ishikawa iterations for uniformly L-Lipschitzian, asymptotically nonexpansive, nonexpansive, and Banach contraction mappings, respectively, in closed convex subsets of Banach spaces.

Our aim in this paper is to establish the equivalence between the convergences of Mann and Ishikawa iteration methods with errors for ϕ-Lipschitzian strongly accretive and ϕ-strongly pseudocontractive operators, respectively, in real Banach spaces. An example is also included to dwell upon the importance of the results obtained. By the way we show that the class of nonexpansive strongly accretive operators in Banach spaces is a proper subset of the class of nonexpansive ϕ-strongly accretive operators.

Lemma 1.1. (see [15]) *Suppose that* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are two sequences of nonnegative numbers satisfying the inequality* $a_{n+1} \le a_n + b_n$ *for* $n \ge 0$. *If* $\sum_{n=0}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 1.2. (see [3], [7]) *Suppose that* $\phi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function with* $\phi(0) = 0$. *If* $\{r_n\}_{n=0}^{\infty}$, $\{s_n\}_{n=0}^{\infty}$, $\{k_n\}_{n=0}^{\infty}$ *and* $\{t_n\}_{n=0}^{\infty}$ *are sequences of nonnegative numbers satisfying the following conditions:*

$$\sum_{n=0}^{\infty} k_n < \infty, \quad \sum_{n=0}^{\infty} t_n < \infty, \quad \sum_{n=0}^{\infty} s_n = \infty, \tag{1.9}$$

$$r_{n+1} \le (1+k_n)r_n - s_n r_n \frac{\phi(r_{n+1})}{1+\phi(r_{n+1}) + r_{n+1}} + t_n, \ n \ge 0, \tag{1.10}$$

then $\lim_{n\to\infty} r_n = 0$.

Lemma 1.3. (see [3], [5], [7]) *Suppose that* X *is a real Banach space and* $T : X \to X$ *is a continuous* ϕ*-strongly accretive operator. Then the equation* $Tx = f$ *has a unique solution for any* $f \in X$.

2. Main Results

Theorem 2.1. *Let* X *be a real Banach space and* $T : X \to X$ *be a Lipschitzian* ϕ*-strongly accretive operator. For a given* $f \in X$, *define a mapping* $S : X \to X$ *by* $Sx = f + x - Tx$ *for all* $x \in X$. *Let the sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{u_n\}_{n=0}^{\infty}$ *be defined by* (1.2) *and* (1.3), *respectively, and* (1.4)-(1.6) *hold. Then, for* $u_0 = x_0 \in X$, *the following assertions are equivalent:*

(a) *the Mann iteration sequence with errors* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$;

(b) *the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$.

Proof. It follows from Lemma 1.3 that the equation $Tx = f$ has a unique solution $x^* \in X$. Then x^* is also a unique fixed point of S. Thus (a) follows from (b) by setting $a'_n = 1$ and $b'_n = c'_n = 0$ for $n \ge 0$.

Let L' denote the Lipschitz constant of T. Clearly, S is a Lipschitz mapping with the Lipschitz constant $L = 1+L'$. Since T is ϕ-strongly accretive, it follows that for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned}\langle (I-S)x - (I-S)y, j(x-y)\rangle &= \langle Tx - Ty, j(x-y)\rangle \\ &\ge \phi(\|x-y\|)\|x-y\| \ge \frac{\phi(\|x-y\|)}{1+\phi(\|x-y\|)+\|x-y\|}\|x-y\|^2\end{aligned}$$

$$= A(x,y)\|x-y\|^2,$$

where

$$A(x,y) = \frac{\phi(\|x-y\|)}{1+\phi(\|x-y\|)+\|x-y\|} \in [0,1) \quad \text{for } x,y \in X.$$

This implies that

$$\langle (I-S-A(x,y))x-(I-S-A(x,y))y, j(x-y)\rangle \geq 0 \quad \text{for } x,y \in X.$$

It follows from Lemma 1.1 of Kato [2] that

$$\|x-y\| \leq \|x-y+r[(I-S-A(x,y))x-(I-S-A(x,y))y]\| \tag{2.1}$$

for $x, y \in X$ and $r > 0$.

To prove that (a) implies that (b), it is necessary to verify that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Since $\|u_n - x^*\| \to 0$ as $n \to \infty$, it is clear that $\{u_n\}_{n=0}^{\infty}$ and $\{Su_n\}_{n=0}^{\infty}$ are both bounded. Note that $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are bounded sequences in X. It follows that

$$\sup_{n\geq 0}\{\|Su_n-u_n\|, \|\sigma_n-u_n\|, \|\delta_n-u_n\|\} \leq M \tag{2.2}$$

for some constant $M > 0$. Set $d_n = b_n + c_n$ and $d'_n = b'_n + c'_n$ for $n \geq 0$. Using (1.2), we get that

$$\begin{aligned} x_n = x_{n+1} &+ d_n x_n - d_n S y_n + c_n(Sy_n - \sigma_n) \\ &= (1+d_n)x_{n+1} + d_n(I-S-A(x_{n+1},u_{n+1}))x_{n+1} \\ &\quad - (1-A(x_{n+1},u_{n+1}))d_n x_n \\ &+ (2-A(x_{n+1},u_{n+1}))d_n^2(x_n - Sy_n) + d_n(Sx_{n+1} - Sy_n) \\ &\quad + c_n[1+(2-A(x_{n+1},u_{n+1}))d_n](Sy_n - \sigma_n). \end{aligned} \tag{2.3}$$

In view of (1.4), we have

$$\begin{aligned} u_n = u_{n+1} &+ d_n u_n - d_n S u_n + c_n(Su_n - \sigma_n) \\ &= (1+d_n)u_{n+1} + d_n(I-S-A(x_{n+1},u_{n+1}))u_{n+1} \\ &\quad - (1-A(x_{n+1},u_{n+1}))d_n u_n \\ &+ (2-A(x_{n+1},u_{n+1}))d_n^2(u_n - Su_n) + d_n(Su_{n+1} - Su_n) \\ &\quad + c_n[1+(2-A(x_{n+1},u_{n+1}))d_n](Su_n - \sigma_n). \end{aligned} \tag{2.4}$$

It follows from (2.1)-(2.4) that

$$\begin{aligned}\|x_n - u_n\| \geq{}& (1+d_n)\|x_{n+1} - u_{n+1} + \frac{d_n}{1+d_n}[(I - S - A(x_{n+1}, u_{n+1}))x_{n+1} \\ &- (I - S - A(x_{n+1}, u_{n+1}))u_{n+1}]\| \\ &- d_n(1 - A(x_{n+1}, u_{n+1}))\|x_n - u_n\| \\ &- (2 - A(x_{n+1}, u_{n+1}))d_n^2\|x_n - Sy_n - u_n + Su_n\| \\ &- d_n\|Sx_{n+1} - Sy_n - Su_{n+1} + Su_n\| \\ &- c_n[1 + (2 - A(x_{n+1}, u_{n+1}))d_n]\|Sy_n - Su_n\| \\ \geq{}& (1+d_n)\|x_{n+1} - u_{n+1}\| - d_n(1 - A(x_{n+1}, u_{n+1}))\|x_n - u_n\| \\ &- 2d_n^2\|x_n - u_n\| - [2d_n^2 + c_n(1+2d_n)]L\|y_n - u_n\| \\ &- d_nL\|x_{n+1} - y_n\| - d_nL\|u_{n+1} - u_n\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \leq{}& \frac{1 + (1 - A(x_{n+1}, u_{n+1}))d_n}{1+d_n}\|x_n - u_n\| \\ &+ 2d_n^2\|x_n - u_n\| + [2d_n^2 + c_n(1+2d_n)]L\|y_n - u_n\| \\ &+ d_nL\|x_{n+1} - y_n\| + d_n(d_n + c_n)LM \\ \leq{}& (1 - A(x_{n+1}, u_{n+1})d_n)\|x_n - u_n\| + 3d_n^2\|x_n - u_n\| \\ &+ (2d_n^2 + 3c_n)L\|y_n - u_n\| + d_nL\|x_{n+1} - y_n\| + 2d_n^2LM. \qquad (2.5)\end{aligned}$$

In light of (1.2) and (2.2), we deduce that

$$\begin{aligned}\|y_n - u_n\| &= \|a_n'x_n + b_n'Sx_n + c_n'\delta_n - u_n\| \\ &\leq a_n'\|x_n - u_n\| + b_n'\|Sx_n - u_n\| + c_n'\|\delta_n - u_n\| \\ &\leq (a_n' + b_n'L)\|x_n - u_n\| + (b_n' + c_n')M \\ &\leq L\|x_n - u_n\| + d_n'M. \qquad (2.6)\end{aligned}$$

In terms of (1.3), (2.2) and (2.6), we infer that

$$\begin{aligned}&\|x_{n+1} - y_n\| \\ = {}& \|(1-d_n)x_n + b_nSy_n + c_n\sigma_n - y_n\| \\ \leq {}& (1-d_n)\|x_n - y_n\| + b_n\|Sy_n - y_n\| + c_n\|\sigma_n - y_n\| \\ \leq {}& (1-d_n)\|(1-d_n')x_n + b_n'Sx_n + c_n'\delta_n - x_n\| \\ &+ b_n\|Sy_n - Su_n\| + b_n\|Su_n - u_n\| + b_n\|y_n - u_n\|\end{aligned}$$

$$\begin{aligned}
&+c_n\|\sigma_n - u_n\| + c_n\|y_n - u_n\| \\
\leq\ & (1-d_n)b'_n(L+1)\|x_n - u_n\| + (1-d_n)b'_n\|Su_n - u_n\| \\
&+(1-d_n)c'_n\|u_n - \delta_n\| + (1-d_n)c'_n\|x_n - u_n\| \\
&+(b_nL + d_n)\|y_n - u_n\| + d_nM \\
\leq\ & (1-d_n)(b'_nL + d'_n)\|x_n - u_n\| + (b_nL + d_n)L\|x_n - u_n\| \\
&+[(1-d_n)d'_n + d_n + (b_nL + d_n)d'_n]M \\
\leq\ & (d'_n + d_nL)(L+1)\|x_n - u_n\| + [d'_n(L+1) + d_n]M. \qquad (2.7)
\end{aligned}$$

Substituting (2.7) and (2.6) into (2.5), we know that

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\| \\
\leq\ & (1 - A(x_{n+1}, u_{n+1})d_n)\|x_n - u_n\| + 3d_n^2\|x_n - u_n\| \\
&+(2d_n^2 + 3c_n)L^2\|x_n - u_n\| + (2d_n^2 + 3c_n)LM \\
&+d_n(d'_n + d_nL)(L+1)L\|x_n - u_n\| \\
&+d_n[d'_n(L+1) + d_n]LM + 2d_n^2LM \\
\leq\ & (1 - A(x_{n+1}, u_{n+1})d_n)\|x_n - u_n\| \\
&+M_1(d_n^2 + d_nd'_n + c_n)\|x_n - u_n\| + M_2(d_n^2 + d_nd'_n + c_n)
\end{aligned}$$

for some constants $M_1 > 0$ and $M_2 > 0$. Set

$$\begin{aligned}
r_n &= \|x_n - u_n\|, \quad s_n = d_n, \\
k_n &= M_1(d_n^2 + d_nd'_n + c_n), \quad t_n = M_2(d_n^2 + d_nd'_n + c_n).
\end{aligned}$$

It follows from (1.5), (1.6) and Lemma 1.2 that $\|x_n - u_n\| \to 0$ as $n \to \infty$ Consequently, we immediately conclude that

$$\|x_n - x^*\| \leq \|u_n - x^*\| + \|u_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Hence the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Tx = f$. This completes the proof. □

The proof of the following theorem goes in a similar fashion as that of Theorem 2.1, so we omit the proof.

Theorem 2.2. *Let X be a real Banach space and $T : X \to X$ be a Lipschitzian ϕ-strongly pseudocontractive operator. Define the Mann and Ishikawa iteration sequences with errors $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ by* (1.3) *and* (1.2) *with $S = T$, respectively. Suppose that* (1.4)-(1.6) *hold. Then, for $u_0 = x_0 \in X$, the following assertions are equivalent:*

(c) *the Mann iteration sequence with errors* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T*;*

(d) *the Ishikawa iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T*.*

It follows from Theorem 2.1 and Theorem 2.2 that the following theorem holds true.

Theorem 2.3. *Let* X *be a real Banach space and* $T : X \to X$ *be a Lipschitzian* ϕ*-strongly accretive operator. For a given* $f \in X$*, define a mapping* $S : X \to X$ *by* $Sx = f + x - Tx$ *for all* $x \in X$*. Let the Mann and Ishikawa iteration sequences* $\{u_n\}_{n=0}^{\infty}$ *and* $\{x_n\}_{n=0}^{\infty}$ *be defined by* (1.8) *and* (1.7)*, respectively. Suppose that* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *satisfy the following conditions:*

$$\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty, \quad \sum_{n=0}^{\infty} \alpha_n^2 < \infty, \quad \sum_{n=0}^{\infty} \alpha_n = +\infty. \tag{2.8}$$

Then, for $u_0 = x_0 \in X$*, the following assertions are equivalent:*

(e) *the Mann iteration sequence* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$*;*

(f) *the Ishikawa iteration sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of* $Tx = f$*.*

Theorem 2.4. *Let* X *be a real Banach space and* $T : X \to X$ *be a Lipschitzian* ϕ*-strongly pseudocontractive operator. Let the Mann and Ishikawa iteration sequences* $\{u_n\}_{n=0}^{\infty}$ *and* $\{x_n\}_{n=0}^{\infty}$ *be defined by* (1.8) *and* (1.7) *with* $S = T$*, respectively, and* (2.8) *hold. Then, for* $u_0 = x_0 \in X$*, the following assertions are equivalent:*

(g) *the Mann iteration sequence* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T*;*

(h) *the Ishikawa iteration sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T*.*

Remark 2.1. Rhoades-Soltuz [12, Theorem 4] proved that if K is a closed convex subset of a Banach space and if $T : K \to K$ is a Lipschitzian strongly pseudocontractive mapping, then the Mann and Ishikawa iteration schemes under certain conditions are equivalent. Theorem 2.2 extends the result of Rhoades-Soltuz to both the Mann and Ishikawa iteration methods with errors and the class of Lipschitzian ϕ-strongly pseudocontractive operators. The following example reveals that Theorem 2.2 is an indeed generalization of Theorem 4 of Rhoades-Soltuz [12], and that the class of Lipschitzian strongly pseudocontractive operators in Banach spaces is a proper subset of the class of Lipschitzian ϕ-strongly pseudocontractive operators.

Example 2.1. Let $E = (-\infty, +\infty)$ with the usual norm $|\cdot|$. Define $T : E \to E$ and $\phi : [0, +\infty) \to [0, +\infty)$ by

$$Tx = \begin{cases} \frac{x^2}{1+x} & \text{for } x \in [0, +\infty), \\ x & \text{for } x \in (-\infty, 0) \end{cases} \quad \text{and } \phi(t) = \frac{t^2}{1+t} \quad \text{for } t \in [0, +\infty),$$

respectively. Set

$$\begin{aligned} a_n &= 1 - \frac{1}{3n+3} - \frac{1}{2(n+1)^2}, \quad b_n = \frac{1}{3n+3}, \quad c_n = \frac{1}{2(n+1)^2}, \\ a'_n &= 1 - \frac{2}{3\sqrt{n+2}}, \quad b'_n = c'_n = \frac{1}{3\sqrt{n+2}}, \quad n \geq 0. \end{aligned}$$

In order to prove that T is ϕ-strongly accretive, we have to consider the following possible cases:

Case 1. Suppose that $x, y \in [0, +\infty)$. It follows that

$$\begin{aligned} &\langle Tx - Ty, x - y \rangle - \phi(|x-y|)|x-y| \\ &\quad = \left(\frac{x^2}{1+x} - \frac{y^2}{1+y} \right)(x-y) - \frac{(x-y)^2}{1+|x-y|}|x-y| \\ &\quad\quad = (x-y)^2 \frac{x+y+xy-|x-y|}{(1+x)(1+y)(1+|x-y|)} \geq 0, \end{aligned}$$

which implies that

$$\langle Tx - Ty, x - y \rangle \geq \phi(|x-y|)|x-y|. \tag{2.9}$$

Case 2. Suppose that $x, y \in (-\infty, 0)$. It is easy to verify that

$$\langle Tx - Ty, x - y \rangle = (x-y)^2 \geq \frac{|x-y|^3}{1+|x-y|} = \varphi(|x-y|)|x-y| \,.$$

Case 3. Suppose that $x \in [0, +\infty)$ and $(-\infty, 0)$. Then

$$\begin{aligned} &\langle Tx - Ty, x - y \rangle - \phi(|x-y|)|x-y| \\ &\quad = \left(\frac{x^2}{1+x} - y \right)(x-y) - \frac{(x-y)^3}{1+x-y} = \frac{-y(x-y)}{(1+x)(1+y)(1+x-y)} \geq 0, \end{aligned}$$

which means that (2.8) holds.

Case 4. Suppose that $x \in (-\infty, 0)$ and $y \in [0, +\infty)$. As in the proof of Case 3, we conclude that (2.8) holds.

Now we assert that T is nonexpansive. We consider the following possible cases:

Case 1. Suppose that $x, y \in [0, +\infty)$. Then

$$|Tx - Ty| = \left|\frac{x^2}{1+x} - \frac{y^2}{1+y}\right| = \frac{x+y+xy}{(1+x)(1+y)}|x-y| \leq |x-y|.$$

Case 2. Suppose that $x, y \in (-\infty, 0)$. It is clear that

$$|Tx - Ty| = |x - y|.$$

Case 3. Suppose that $x \in [0, +\infty)$ and $y \in (-\infty, 0)$. It follows that

$$|Tx - Ty| - |x - y| = \frac{x^2}{1+x} - y - (x - y) = -\frac{x}{1+x} \leq 0,$$

that is,

$$|Tx - Ty| \leq |x - y|. \tag{2.10}$$

Case 4. Suppose that $x \in (-\infty, 0)$ and $y \in [0, +\infty)$. In a similar way, we deduce that (2.9) holds.

Consequently, Theorem 2.2 ensures the equivalence of the Mann and the Ishikawa iteration methods with errors for Lipschitzian ϕ-strongly pseudocontractive operator $(I - T)$. But Theorem 4 of Rhoades-Soltuz [12] is not applicable since $(I - T)$ is not strongly pseudocontractive, that is, T is not strongly accretive. In fact, for any given $k \in (0, 1)$, there exist $(x_k, y_k) = \left(\frac{k}{2(1-k)}, 0\right) \in E \times E$ such that

$$\begin{aligned}&\langle Tx_k - Ty_k, x_k - y_k\rangle - k|x_k - y_k|^2 \\ &\qquad = \frac{x_k^3}{1+x_k} - kx_k^2 = x_k^2\left(\frac{x_k}{1+x_k} - k\right) = -kx_k^2\frac{1-k}{2-k} < 0,\end{aligned}$$

which implies that $\langle Tx_k - Ty_k, x_k - y_k\rangle < k|x_k - y_k|^2$.

3. Acknowledgements

This work was supported by Korea Research Foundation Grant (KRF-2003-005-C00013).

References

[1] S.S. Chang, Y.J. Cho, J.K. Kim, The equivalence between the convergence of modified Picard, modified Mann, and modified Ishikawa iterations, *Math. Computer Modelling*, **37** (2003), 985-991.

[2] T. Kato, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan*, **196** (1967), 508-520.

[3] Z. Liu, Y.G. Xu, Y.J. Cho, Iterative solution of nonlinear equations with ϕ-strongly accretive operators, *Arch. Math.*, **77** (2001), 508-516.

[4] Z. Liu, S.M. Kang, Iterative approximation of fixed points for ϕ-hemicontractive operators in arbitrary Banach spaces, *Acta Sci. Math.*, **67** (2001), 821-831.

[5] Z. Liu, S.M. Kang, Convergence theorems for ϕ-strongly accretive and ϕ-hemicontractive operators, *J. Math. Anal. Appl.*, **253** (2001), 35-49.

[6] Z. Liu, S.M. Kang, Stability of Ishikawa iteration methods with errors for strong pseudocontractions and nonlinear equations involving accretive operators in arbitrary real Banach spaces, *Math. Computer Modelling*, **34** (2001), 319-330.

[7] Z. Liu, S.M. Kang, Convergence and stability of the Ishikawa iteration procedures with errors for nonlinear equations of the ϕ-strongly accretive type, *Neural. Parallel & Scientific Computations*, **9** (2001), 103-118.

[8] Z. Liu, M. Bounias, S.M. Kang, Iterative approximation of solutions to nonlinear equations of ϕ-strongly accretive operators in Banach spaces, *Rocky Mountain J. Math.*, **32** (2002), 981-997.

[9] Z. Liu, J.K. Kim, S.M. Kang, Necessary and sufficient conditions for convergence of Ishikawa iterative schemas with errors to ϕ-hemicontractive mappings, *Commun. Korean Math. Soc.*, **18** (2003), 251-261.

[10] Z. Liu, S.M. Kang, Iterative solutions of nonlinear equations with ϕ-strongly accretive operators in uniformly smooth Banach spaces, *Computers Math. Appl.*, **45** (2003), 623-634.

[11] M.O. Osilike, Iterative solution of nonlinear equations of the ϕ-strongly accretive type, *J. Math. Anal. Appl.*, **200** (1996), 259-271.

[12] B.E. Rhoades, Stefan M. Soltuz, On the equivalence of Mann and Ishikawa iteration methods, *Internat. J. Math. Math. Sci.*, **7** (2003), 451-459.

[13] B.E. Rhoades, Stefan M. Soltuz, The equivalence of Mann iteration and Ishikawa iteration for non-Lipschitizan operators, *Internat. J. Math. Math. Sci.*, **42** (2003), 2645-2651.

[14] B.E. Rhoades, Stefan M. Soltuz, The equivalence between the convergences of Ishikawa and Mann iterations for an asymptotically pseudocontractive map, *J. Math. Anal. Appl.*, **283** (2003), 681-688.

[15] X.L. Weng, Fixed point iteration for local strictly pseudo-contractive mappings, *Proc. Amer. Math. Soc.*, **113** (1991), 727-731.

[16] Y. Xu, Ishikawa, Mann iterative processes with errors for nonlinear strongly accretive operator equations, *J. Math. Anal. Appl.*, **224** (1998), 91-101.

[17] Y. Xu, Z. Liu, On estimation and control of errors of the Mann iteration process, *J. Math. Anal. Appl.*, **286** (2003), 804-806.