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THE EQUIVALENCE AMONG MODIFIED MANN ITERATION, MODIFIED ISHIKAWA ITERATION AND MODIFIED MULTISTEP ITERATION

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Abstract: The purpose of this paper is to investigate the convergence of modified Mann iteration and modified Ishikawa iteration are equivalent to the convergence of modified multistep iteration which is introduced by Rhoades and Soltuz in *Nonlinear Anal.*, **58** (2004), 219-228, for some kinds of nonlinear mappings. Furthermore, we show that the equivalence of stability among modified Mann iteration, modified Ishikawa iteration and modified multistep iteration.

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1. Introduction

Throughout this paper, we assume that X is a Banach space, B is a nonempty

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convex subset of X and $T : B \to B$ is a mapping B. The three most popular iteration procedures for obtaining fixed points of T, if they exist, are modified Mann iteration, defined by

$$u_1 \in B, \quad u_{n+1} = (1-\alpha_n)u_n + \alpha_n T^n u_n, \quad \forall n \geq 1, \tag{1.1}$$

modified Ishikawa iteration, defined by

$$\begin{aligned} &z_1 \in B, \quad z_{n+1} = (1-\alpha_n)z_n + \alpha_n T^n y_n, \\ &y_n = (1-\beta_n)z_n + \beta_n T^n z_n, \quad \forall n \geq 1 \end{aligned}$$

and modified three-step iteration procedure which is introduced by Rhoades and Soltuz in [37],

$$\begin{aligned} &v_1 \in B, \quad t_n = (1-\gamma_n)v_n + \gamma_n T^n v_n, \\ &w_n = (1-\beta_n)v_n + \beta_n T^n t_n, \\ &v_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n w_n, \quad \forall n \geq 1 \end{aligned} \tag{1.2}$$

for certain choices of $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$, $\{\gamma_n\}_{n\geq 1} \subset [0,1]$. Replacing T^n by T in (1.1), (1.2) and (1.3), we obtain ordinary Mann, Ishikawa and three-step iteration, respectively.

In 2004, Rhoades and Soltuz in [37] showed us a more general iteration procedure than ordinary iteration procedures which is called multistep iteration of arbitrary fixed order $p \geq 2$, defined by

$$\begin{aligned} &x_1 \in B, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ &y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = 1, \cdots, p-2, \\ &y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \quad \forall n \geq 1. \end{aligned} \tag{1.3}$$

In addition, they proved the convergence of Mann and Ishikawa iterations are equivalent to the convergence of a multistep iteration under certain conditions, moreover, they obtained the equivalence of stability among those iterations listed above.

In the last five decades, numerous papers were published on the iterative approximations of fixed points of various kinds of mappings in metric spaces, Hilbert spaces and Banach spaces in [1]-[37], respectively. Some researchers established several classes of iteration processes such as Mann iteration, Ishikawa iteration, multistep iteration and some modified iterations. In 2003, Chang, Cho and Kim disclosed the equivalence among modified Picard iteration, Modified Mann iteration and modified Ishikawa iterations under some conditions in [2].

Inspired and motivated by these works in [1]-[37], in this paper, we introduce a more general modified iteration procedure of a arbitrary fixed order $p \geq 2$, defined by

$$\begin{array}{l} x_1 \in B, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1, \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T^n y_n^{i+1}, \quad i = 1, \cdots, p-2, \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T^n x_n, \quad \forall n \geq 1, \end{array} \tag{1.4}$$

which is called modified multistep iteration, where $\{\alpha_n\}_{n\geq 1}$ satisfies

$$\{\alpha_n\}_{n\geq 1} \subset (0,1), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \Sigma_{n=1}^{\infty} \alpha_n = \infty, \quad \forall n \geq 1 \tag{1.5}$$

and

$$\{\beta_n^i\}_{n\geq 1} \subset (0,1), \quad 1 \leq i \leq p-1, \quad \lim_{n\to\infty} \beta_n^1 = 0. \tag{1.6}$$

Taking $p = 3$ and $p = 2$ in (1.5), respectively, we obtain iterations (1.3) and (1.2), respectively. Let $F(T)$ denote the set of fixed points of T.

The aim of this paper is to prove the equivalence among the convergence of modified Mann iteration, modified Ishikawa iteration and modified multistep iteration for various classes of mappings. Our results extend, improve and unify the corresponding results in [36], [37].

Definition 1.1. The mapping $T : B \to B$ is said to be:

(1) *asymptotically nonexpansive* in the intermediate sense if T^m is continuous for some positive integer m and

$$\limsup_{n\to\infty} \sup_{x,y\in B} (\|T^n x - T^n y\| - \|x-y\|) \leq 0, \quad \forall n \geq 1; \tag{1.7}$$

(2) *strongly successively pseudocontractive* if there exist $k \in (0,1)$ and $n_0 \geq 1$ such that

$$\|x-y\| \leq \|x - y + t[(I - T^n - kI)x - (I - T^n - kI)y]\| \tag{1.8}$$

for all $x, y \in B$, $t > 0$ and $n \geq n_0$.

Lemma 1.1. (see [3]) *Suppose that* $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ *and* $\{c_n\}_{n\geq 1}$ *are nonnegative sequences such that*

$$a_{n+1} \leq (1-c_n)a_n + b_n c_n, \quad \forall n \geq 1, \tag{1.9}$$

with $\{c_n\}_{n\geq 1} \subset [0,1]$, $\sum_{n=0}^{\infty} c_n = \infty$ *and* $\lim_{n\to\infty} b_n = 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

2. The Asymptotically Nonexpansive Mappings in the Intermediate Sense

In this section, we prove the equivalence between modified Mann iteration and modified multistep iteration for asymptotically nonexpansive mappings in the intermediate sense.

Theorem 2.1. *Let B a closed convex bounded subset of a Banach space X and $\{x_n\}_{n\geq 1}$ and $\{u_n\}_{n\geq 1}$ defined by* (1.5) *and* (1.1)*, respectively, with $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n^i\}_{n\geq 1} \subset (0,1)$, where $i = 1, \cdots, p-1$, satisfying* (1.6) *and* (1.7)*. Let $T : B \to B$ be an asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive mapping. Put*

$$c_n = \max\left\{0, \sup_{x,y\in B} (\|T^n x - T^n y\| - \|x - y\|)\right\}, \quad \forall n \geq 1, \tag{2.1}$$

so that

$$\lim_{n\to\infty} c_n = 0. \tag{2.2}$$

If $u_1 = x_1 \in B$, then the following two assertions are equivalent:

(a) *Modified Mann iteration* (1.1) *converges to $x^* \in F(T)$;*

(b) *Modified multistep iteration* (1.5) *converges to $x^* \in F(T)$.*

Proof. If the modified multistep iteration (1.5) converges to $x^* \in F(T)$, by setting $\beta_n^i = 0$ for all $n \geq 1$, $1 \leq i \leq p-1$, in (1.5), we obtain the convergence of modified Mann iteration (1.1). Conversely, we shall prove that the convergence of modified Mann iteration implies the convergence of modified multistep iteration. From (1.5) and (1.1), we have that

$$\begin{aligned} x_n = x_{n+1} + \alpha_n x_n - \alpha_n T^n y_n^1 \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - kI)x_{n+1} \\ &- (1-k)\alpha_n x_n + (2-k)\alpha_n^2(x_n - T^n y_n^1) + \alpha_n(T^n x_{n+1} - T^n y_n^1), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} u_n = (1+\alpha_n)u_{n+1} + \alpha_n(I - T^n - kI)u_{n+1} - (1-k)\alpha_n u_n \\ + (2-k)\alpha_n^2(u_n - T^n u_n) + \alpha_n(T^n u_{n+1} - T^n u_n), \end{aligned} \tag{2.4}$$

for all $n \geq 1$. From (2.3), (2.4) and (1.9) to obtain that

$$\begin{aligned}
\|x_n - u_n\| = \|&(1+\alpha_n)(x_{n+1} - u_{n+1}) \\
&+ \alpha_n[(I - T^n - kI)x_{n+1} - (I - T^n - kI)u_{n+1}] \\
&- (1-k)\alpha_n(x_n - u_n) + (2-k)\alpha_n^2[x_n - u_n - (T^n y_n^1 - T^n u_n)] \\
&+ \alpha_n[T^n x_{n+1} - T^n y_n^1 - (T^n u_{n+1} - T^n u_n)]\| \\
&\geq (1+\alpha_n)\|(x_{n+1} - u_{n+1}) + \frac{\alpha_n}{1+\alpha_n}[(I - T^n - kI)x_{n+1} \\
&- (I - T^n - kI)u_{n+1}]\| - (1-k)\alpha_n\|x_n - u_n\| \\
&- (2-k)\alpha_n^2\|x_n - u_n - (T^n y_n^1 - T^n u_n)\| \\
&- \alpha_n\|T^n x_{n+1} - T^n y_n^1 - (T^n u_{n+1} - T^n u_n)\| \\
&\geq (1+\alpha_n)\|x_{n+1} - u_{n+1}\| - (1-k)\alpha_n\|x_n - u_n\| \\
&- (2-k)\alpha_n^2\|x_n - u_n - (T^n y_n^1 - T^n u_n)\| \\
&- \alpha_n\|T^n x_{n+1} - T^n y_n - (T^n u_{n+1} - T^n u_n)\|, \quad \forall n \geq 1. \qquad (2.5)
\end{aligned}$$

Obviously, we get that

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\leq (1 + (1-k)\alpha_n)(1 - \alpha_n + \alpha_n^2)\|x_n - u_n\| \\
&+ \alpha_n[(2-k)\alpha_n\|x_n - T^n y_n^1\| + (2-k)\alpha_n\|u_n - T^n u_n\| \\
&+ \|T^n u_{n+1} - T^n u_n\| + \|T^n x_{n+1} - T^n y_n^1\|] \\
&= (1 + (1-k)\alpha_n)(1 - \alpha_n + \alpha_n^2)\|x_n - u_n\| + \alpha_n\sigma_n \qquad (2.6)
\end{aligned}$$

for all $n \geq 1$, where

$$\begin{aligned}
\sigma_n := (2-k)\alpha_n\|x_n - T^n y_n^1\| &+ (2-k)\alpha_n\|u_n - T^n u_n\| \\
&+ \|T^n u_{n+1} - T^n u_n\| + \|T^n x_{n+1} - T^n y_n^1\|. \qquad (2.7)
\end{aligned}$$

Obviously, $\{x_n\}_{n\geq 1} \subset B$ and $\{x_n\}_{n\geq 1}$ is a bounded sequence. Set

$$\begin{aligned}
M := \sup\{\|x_n\|, \|T^n x_n\|, \|T^n y_n^i\|, \|u_n\|, \|T^n u_n\| : n \geq 1, \\
1 \leq i \leq p-1\}. \qquad (2.8)
\end{aligned}$$

There is no doubt that $M < \infty$. Note that

$$\|u_n - T^n u_n\| \leq \|u_n\| + \|T^n u_n\| \leq 2M, \quad \forall n \geq 1. \qquad (2.9)$$

It is clear that $\{\|T^n u_{n+1} - T^n u_n\|\}_{n\ge 1}$ converges to zero, because

$$\begin{aligned} 0 &\le \|T^n u_{n+1} - T^n u_n\| \\ &= \|T^n u_{n+1} - T^n u_n\| - \|u_{n+1} - u_n\| + \|u_{n+1} - u_n\| \\ &\le c_n + \|u_{n+1} - u_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} \|y_n^1 - x_{n+1}\| &= \| - \beta_n^1 x_n + \beta_n^1 T^n y_n^2 + \alpha_n x_n - \alpha_n T^n y_n^1\| \\ &\le \alpha_n(\|x_n\| + \|T^n y_n^1\|) + \beta_n^1(\|x_n\| + \|T^n y_n^2\|) \\ &\le (\alpha_n + \beta_n^1) 2M \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.11}$$

we obtain that

$$\begin{aligned} \|T^n x_{n+1} - T^n y_n^1\| &= (\|T^n y_n^1 - T^n x_{n+1}\| - \|y_n^1 - x_{n+1}\|) + \|y_n^1 - x_{n+1}\| \\ &\le c_n + \|y_n^1 - x_{n+1}\| \to 0 \quad as \quad n \to \infty. \end{aligned} \tag{2.12}$$

From (2.9), (2.10), (2.12) and $\lim_{n\to\infty} \alpha_n = 0$, we know that $\lim_{n\to\infty} \sigma_n = 0$.

Notice that $\lim_{n\to\infty} \alpha_n = 0$. It follow that there exists $N \ge 1$ such that

$$\alpha_n \le \frac{k}{2}, \quad \forall n \ge N. \tag{2.13}$$

Obviously,

$$(1 + (1-k)\alpha_n)(1 - \alpha_n + \alpha_n^2) \le 1 - k\alpha_n + \alpha_n^2 = 1 - \frac{k}{2}\alpha_n, \quad \forall n \ge N. \tag{2.14}$$

It follows from (2.14) and (2.6) that

$$\|x_{n+1} - u_{n+1}\| \le (1 - \frac{k}{2}\alpha_n)\|x_n - u_n\| + \alpha_n \sigma_n, \quad \forall n \ge N. \tag{2.15}$$

Using (2.16), (2.15), (1.10) and Lemma 1.1 we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0,$$

so that

$$0 \le \|x_n - x^*\| \le \|x_n - u_n\| + \|u_n - x^*\| \to 0 \quad \text{as } n \to \infty.$$

Hence $\lim_{n\to\infty} \|x_n - x^*\| = 0$. This completes the proof. $\square$

Remark 2.1. Theorem 2.1 is a generalization of Theorem 4 in [36].

According to Theorem 2.1, we can get the following result.

Corollary 2.1. *Let B a closed convex bounded subset of a Banach space X and $\{x_n\}_{n\geq 1}$ and $\{u_n\}_{n\geq 1}$ defined by* (1.5) *and* (1.1)*, respectively. Let $T : B \to B$ be an asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive mapping. Suppose that $\{c_n\}_{n\geq 1}$ is as in Theorem 2.1. If $u_1 = x_1 \in B$, then the following assertions are equivalent:*

(a) *Modified Mann iteration* (1.1) *converges to $x^* \in F(T)$.*
(b) *Modified Ishikawa iteration* (1.2) *converges to $x^* \in F(T)$.*
(c) *Modified three-step iteration* (1.3) *converges to $x^* \in F(T)$.*
(d) *Modified multistep iteration* (1.5) *converges to $x^* \in F(T)$.*

3. The Strong Successively Pseudocontractive Mappings

Let X be a real Banach space, B be a nonempty subset of X and $T : B \to B$. The mapping $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X$, is called the normalized duality mapping, where X^* denotes the dual space of X and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. The Hahn-Banach Theorem assures that $Jx \neq \emptyset, \forall x \in X$. It is obvious that $\langle j(x), y\rangle \leq \|x\|\|y\|, \forall x, y \in X,\ j(x) \in J(x)$.

Definition 3.1. The mapping $T : B \to B$ is called *strongly successively pseudocontractive mapping* if there exist $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k\|x - y\|^2, \quad \forall x, y \in B. \tag{3.1}$$

The following lemmas play a crucial role in the proofs of our main results.

Lemma 3.1. (see [1]) *Let X be a Banach space. Then the following conditions are equivalent:*

(a) *X is uniformly smooth.*
(b) *X^* is uniformly convex.*
(c) *J is a single valued and uniformly continuous on any bounded subset of X.*

Lemma 3.2. (see [32]) *If X is a real Banach space, then the following relation is true:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in X,\ j(x + y) \in J(x + y). \tag{3.2}$$

We are able to prove the following result.

Theorem 3.1. *Let X be a real Banach space with dual uniformly convex and B a nonempty closed convex bounded subset of X. Let $T : B \to B$ be a strongly successively pseudocontractive mapping and $\{x_n\}_{n\geq 1}, \{u_n\}_{n\geq 1}$ defined by* (1.5) *and* (1.1) *with $\{\alpha_n\}_{n\geq 1}, \{\beta_n^i\}_{n\geq 1} \subset (0,1), i = 1, \cdots, p-1$, satisfying* (1.6) *and* (1.7). *Then for $u_1 = x_1 \in B$, the following assertions are equivalent:*

(a) *Modified Mann iteration* (1.1) *converges to $x^* \in F(T)$.*

(b) *Modified multistep iteration* (1.5) *converges to $x^* \in F(T)$.*

Proof. Using (1.1), (1.5) and (3.2), we get that

$$
\begin{aligned}
& \|x_{n+1} - u_{n+1}\|^2 \\
= \ & \|(1-\alpha_n)(x_n - u_n) + \alpha_n(T^n y_n^1 - T^n u_n)\|^2 \\
\leq \ & (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle T^n y_n^1 - T^n u_n, J(x_{n+1} - u_{n+1})\rangle \\
= \ & (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle T^n y_n^1 - T^n u_n, J(y_n^1 - u_n)\rangle \\
& +2\alpha_n \langle T^n y_n^1 - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n)\rangle \\
\leq \ & (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|y_n^1 - u_n\|^2 \\
& +2\alpha_n \langle T^n y_n^1 - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n)\rangle \\
\leq \ & (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|y_n^1 - u_n\|^2 \\
& +2\alpha_n \|T^n y_n^1 - T^n u_n\| \|J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n)\| \\
\leq \ & (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|y_n^1 - u_n\|^2 \\
& +4\alpha_n M \|J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n)\|, \quad \forall n \geq 1,
\end{aligned} \tag{3.3}
$$

where M satisfies (2.8). Notice that

$$
\begin{aligned}
& \|(x_{n+1} - u_{n+1}) - (y_n^1 - u_n)\| \\
= \ & \| -\alpha_n x_n + \alpha_n T^n y_n^1 + \beta_n^1 x_n - \beta_n^1 T^n y_n^2 + \alpha_n u_n - \alpha_n T^n u_n\| \\
\leq \ & \alpha_n(\|x_n\| + \|T^n y_n^1\| + \|u_n\| + \|T^n u_n\|) + \beta_n^1(\|x_n\| + \|T^n y_n^2\|) \\
\leq \ & 4(\alpha_n + \beta_n^1)M \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}
$$

and J is single valued and uniformly continuous on every bounded set. It follows that

$$
J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}
$$

From (1.5) and (3.2), we get that

$$\begin{aligned} & \|y_n^1 - u_n\|^2 \\ = \; & \|(1-\beta_n^1)(x_n - u_n) + \beta_n^1(T^n y_n^2 - u_n)\| \\ \le \; & (1-\beta_n^1)^2\|x_n - u_n\|^2 + 2\beta_n^1\langle T^n y_n^2 - u_n, J(y_n^1 - u_n)\rangle \\ \le \; & \|x_n - u_n\|^2 + \beta_n^1\|T^n y_n^2 - u_n\|\|y_n^1 - u_n\| \\ \le \; & \|x_n - u_n\|^2 + \beta_n^1 M_2, \quad \forall n \ge 1, \end{aligned} \tag{3.5}$$

where $M_2 := \sup\{\|T^n y_n^2 - u_n\|\|y_n^1 - u_n\| : n \ge 1\} < \infty$. Replacing (3.5) in (3.3), we obtain that

$$\begin{aligned} \|x_{n+1} - u_{n+1}\|^2 \le \; & (1 - 2(1-k)\alpha_n + \alpha_n^2)\|x_n - u_n\|^2 + 2\alpha_n k\beta_n^1 M_2 \\ & + 4\alpha_n M\|J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n)\| \end{aligned} \tag{3.6}$$

for any $n \ge 1$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exists $n_0 \ge 1$ satisfying

$$\alpha_n \le 1 - k, \quad \forall n \ge n_0. \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain that

$$\begin{aligned} & \|x_{n+1} - u_{n+1}\|^2 \\ \le \; & (1 - (1-k)\alpha_n)\|x_n - u_n\|^2 + 2\alpha_n k\beta_n^1 M_2 \\ & + 4\alpha_n M\|J(x_{n+1} - u_{n+1}) - J(y_n^1 - u_n)\|, \quad \forall n \ge n_0. \end{aligned} \tag{3.8}$$

From (3.4), (3.8) and Lemma 1.1, we get that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.9}$$

Suppose that modified Mann iteration converges, that is, $\lim_{n\to\infty} u_n = x^*$. It follows from

$$0 \le \|x^* - x_n\| \le \|u_n - x^*\| + \|x_n - u_n\|, \quad \forall n \ge 1$$

and (3.9) that $\lim_{n\to\infty} x_n = x^*$. Analogously $\lim_{n\to\infty} x_n = x^*$ implies $\lim_{n\to\infty} u_n = x^*$. This completes the proof. □

Remark 3.1. Theorem 3.1 generalizes Theorem 8 in [36].

From Theorem 3.1, we get Corollary 3.1 listed below.

Corollary 3.1. *Let X be a real Banach space with dual uniformly convex and B a nonempty closed convex bounded subset of X. Let $T : B \to B$ be a strongly successively pseudocontractive mapping and $\{x_n\}_{n\geq 1}, \{u_n\}_{n\geq 1}$ defined by* (1.5) *and* (1.1), *respectively. Then for $u_1 = x_1 \in B$, the following assertions are equivalent:*

(a) *Modified Mann iteration* (1.1) *converges to $x^* \in F(T)$.*
(b) *Modified Ishikawa iteration* (1.2) *converges to $x^* \in F(T)$.*
(c) *Modified three-step iteration* (1.3) *converges to $x^* \in F(T)$.*
(d) *Modified multistep iteration* (1.5) *converges to $x^* \in F(T)$.*

4. The Equivalence between T-Stability

In this section, we give the equivalence of stability among modified Mann iteration, modified Ishikawa iteration and modified multistep iteration for various classes of mappings.

Let $x^* \in F(T)$. We consider two sequences $\{\varepsilon_n\}_{n\geq 1}$ and $\{\delta_n\}_{n\geq 1}$, where

$$\varepsilon_n := \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n^1\| \tag{4.1}$$

and

$$\delta_n := \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \tag{4.2}$$

for all $n \geq 1$.

Definition 4.1. If $\lim\limits_{n\to\infty} \varepsilon_n = 0$ (respectively, $\lim\limits_{n\to\infty} \delta_n = 0$) implies the fact that $\lim\limits_{n\to\infty} x_n = x^*$(respectively $\lim\limits_{n\to\infty} u_n = x^*$), then (1.5) (respectively, (1.1)) is said to be T-*stable.*

Lemma 4.1. *Let X be a normed space with B a nonempty convex closed and bounded subset. Let $T : B \to B$ be a mapping. If the modified multistep* (*respectively, Mann*) *iteration converges, then $\lim\limits_{n\to\infty} \varepsilon_n = 0$* (*respectively, $\lim\limits_{n\to\infty} \delta_n = 0$*).

Proof. Let $\lim\limits_{n\to\infty} x_n = x^*$. Since B is bounded, then $\{x_n\}_{n\geq 1}$ and $\{T^n y_n^1\}_{n\geq 1}$ are bounded. It follows from $\lim\limits_{n\to\infty} \alpha_n = 0$ and (4.1) that

$$\begin{aligned} 0 \ \leq \varepsilon_n \ &\leq \|x_{n+1} - x_n\| + \alpha_n\|x_n - T^n y_n^1\| \\ &\leq \ \|x_{n+1} - x_n\| + \alpha_n(\|x_n\| + \|T^n y_n^1\|) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

that is, $\lim\limits_{n\to\infty} \varepsilon_n = 0$. This completes the proof. □

According to Lemma 4.1, we are able to prove the following result.

Theorem 4.1. *Let B a closed convex bounded subset of a Banach space X and $\{x_n\}_{n\geq 1}$ and $\{u_n\}_{n\geq 1}$ defined by* (1.5) *and* (1.1), *respectively, with $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n^i\}_{n\geq 1} \subset (0,1), i = 1, \cdots, p-1$, satisfying* (1.6) *and* (1.7). *Let $T : B \to B$ be an asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive mapping. Let $\{c_n\}_{n\geq 1}$ be as in* (2.1) *satisfying $\lim\limits_{n\to\infty} c_n = 0$. If $u_1 = x_1 \in B$, then the following two assertions are equivalent:*

(a) *Modified multistep iteration* (1.5) *is T-stable.*
(b) *Modified Mann iteration* (1.1) *is T-stable.*

Proof. By Definition 4.1 we know that the equivalence $(a) \Leftrightarrow (b)$ means that $\lim\limits_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim\limits_{n\to\infty} \delta_n = 0$. The implication $\lim\limits_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim\limits_{n\to\infty} \delta_n = 0$ is obvious by setting $\beta_n^i = 0, i = 1, \cdots, p-1$, in (1.5). Suppose that (1.1) is T-stable. Using Definition 4.1, we get that $\lim\limits_{n\to\infty} \delta_n = 0 \Rightarrow \lim\limits_{n\to\infty} u_n = x^*$. Notice that Theorem 2.1 assures that $\lim\limits_{n\to\infty} u_n = x^* \Rightarrow \lim\limits_{n\to\infty} x_n = x^*$. In view of Lemma 4.1 we deduce that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Consequently, we easily see that $\lim\limits_{n\to\infty} \delta_n = 0 \Rightarrow \lim\limits_{n\to\infty} \varepsilon_n = 0$. This completes the proof. □

From Theorem 3.1 and Lemma 4.1, we get the following result.

Theorem 4.2. *Let X be a real Banach space with dual uniformly convex and B a nonempty closed convex bounded subset of X. Let $T : B \to B$ be a strongly successively pseudocontractive mapping and $\{x_n\}_{n\geq 1}$, $\{u_n\}_{n\geq 1}$ defined by* (1.5) *and* (1.1), *respectively, with $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n^i\}_{n\geq 1} \subset (0,1), i = 1, \cdots, p-1$, satisfying* (1.6) *and* (1.7). *Then for $u_1 = x_1 \in B$, the following assertions are equivalent:*

(a) *Modified multistep iteration* (1.5) *is T-stable.*
(b) *Modified Mann iteration* (1.1) *is T-stable.*

Remark 4.1. Theorem 11 (respectively, Theorem 13) in [36] is a special case of Theorem 4.1 (respectively, Theorem 4.2).

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