LINEAR INITIAL-VALUE PROBLEMS
COUNTABLY DETERMINED

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Abstract: In this paper we approximate the solution of a linear initial-value problem, making use of a Schauder basis for certain Banach space associated with such a differential problem.

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1. Introduction

In this work some results are discussed in order to approximate the solution of the following initial-value problem: given $x_0 \in \mathbb{R}^n$, $a \in C([\alpha, \alpha + \beta], \mathcal{M}_n(\mathbb{R}))$ ($\mathcal{M}_n(\mathbb{R})$ is the set of all $n \times n$ real matrices) and $b \in C([\alpha, \alpha + \beta], \mathbb{R}^n)$, find $x \in C^1([\alpha, \alpha + \beta], \mathbb{R}^n)$ such that
\begin{equation}
\begin{cases}
x'(t) = a(t)x(t) + b(t), & t \in [\alpha, \alpha + \beta], \\
x(\alpha) = x_0.
\end{cases}
\end{equation}

The key idea is to combine classical techniques (fixed point) with certain properties of Schauder bases. Thus we do not need to solve systems of algebraical
linear equations – collocation methods – or to use quadrature formulas.

Let us recall (Megginson [3]) that a sequence \( \{x_j\}_{j \geq 1} \) in a Banach space \( X \) is said to be a Schauder basis provided that for all \( x \in X \) there exists a unique sequence of scalars \( \{\lambda_j\}_{j \geq 1} \) in such a way that \( x = \sum_{j \geq 1} \lambda_j x_j \). The \( j \)-th (continuous and linear) biorthogonal functional \( x_j^* \) is defined at such an \( x \) as \( x_j^*(x) = \lambda_j \), and the \( j \)-th (continuous and linear) projection \( Q_j \) by \( Q_j(x) = \sum_{i=1}^j \lambda_i x_i \).

Starting in Section 2 with a background about the classical Schauder basis in the Banach space \( C([\alpha, \alpha + \beta]) = C([\alpha, \alpha + \beta], \mathbb{R}) \) and some fixed point questions for the operator related to the initial-value problem considered, we get in Section 3 to an operative expression of such an operator in terms of certain linear combinations of evaluations of the data functions. Such expression enables us to derive, with the Banach Fixed Point Theorem, a result (Theorem 3) to approximate the solution of the initial-value problem. A numerical example in Section 4 illustrates the preceding results.

2. Preliminaries

We begin by introducing the classical Schauder basis for the space \( C([\alpha, \alpha + \beta]) \), endowed with its usual sup-norm. Suppose that \( \{t_j\}_{j \geq 1} \) is a dense sequence of distinct points in \([\alpha, \alpha + \beta]\) such that \( t_1 = \alpha \) and \( t_2 = \alpha + \beta \). The classical Schauder basis \( \{\Gamma_j\}_{j \geq 1} \) (associated with \( \{t_j\}_{j \geq 1} \)) for the Banach space \( C([\alpha, \alpha + \beta]) \) is defined as follows:

\[
\Gamma_1(t) = 1 \quad (\alpha \leq t \leq \alpha + \beta)
\]

and for all \( j > 1 \), \( \Gamma_j \) is the piecewise linear continuous function with nodes at \( t_1, \ldots, t_j \), such that

\[
\Gamma_j(t_i) = 0, \quad \text{for all } 1 \leq i < j, \quad \Gamma_j(t_i) = 1.
\]

In what follows, \( \{\Gamma_j\}_{j \geq 1} \) will denote such basis and \( \{\Gamma_j^*\}_{j \geq 1} \) and \( \{Q_j\}_{j \geq 1} \), respectively, the associated sequences of biorthogonal functionals and projections. In the next statement we collect some basic elementary facts that will play a fundamental role in our results. For a proof, see Megginson [3] or Semadeni [4].

**Proposition 1.** Let \( x \in C([\alpha, \alpha + \beta]) \). Then

\[
\Gamma_1^*(x) = x(t_1)
\] (2.1)
and for all \( j > 1 \),

\[
\Gamma_j^*(x) = x(t_j) - \sum_{i=1}^{j-1} \Gamma_i^*(x) \Gamma_i(t_j). 
\] (2.2)

In particular, for all \( j \geq 1 \) and for all \( i \leq j \),

\[
(Q_j x)(t_i) = x(t_i). 
\] (2.3)

Another tool that we shall use in the following is the Banach Fixed Point Theorem in the following form (Jamenson [2]): let \( T \) be a self-mapping of a (nonempty) Banach space \((X, \| \cdot \|)\) and let \( \{\lambda_j\}_{j \geq 1} \) be a sequence of non-negative real numbers such that the series \( \sum_{j \geq 1} \lambda_j \) is convergent and for all \( x, y \in X \) and for all \( j \geq 1 \),

\[
\|T^j x - T^j y\| \leq \lambda_j \|x - y\|.
\] Then \( T \) has a unique fixed point \( u \in X \). Moreover, if \( \bar{x} \) is an element in \( X \) then

\[
u = \lim_{j \to \infty} T^j (\bar{x}).
\] In fact, we have that for all \( j \geq 1 \),

\[
\|T^j \bar{x} - u\| \leq \left( \sum_{i=j}^{\infty} \lambda_i \right) \|T \bar{x} - \bar{x}\|.
\]

Let us now consider the operator

\[
T : C([\alpha, \alpha + \beta], \mathbb{R}^n) \rightarrow C([\alpha, \alpha + \beta], \mathbb{R}^n),
\]

associated with the initial-value problem (1.1), defined by

\[
(T x)(t) := x_0 + \int_{\alpha}^{t} (a(s)x(s) + b(s)) ds,
\]

\[
(t \in [\alpha, \alpha + \beta], \ x \in C([\alpha, \alpha + \beta], \mathbb{R}^n)), \quad \text{(2.4)}
\]

where the norm in the space \( C([\alpha, \alpha + \beta], \mathbb{R}^n) \) is the sup-sup one:

\[
\|x\|_{\infty} := \sup_{t \in [\alpha, \alpha + \beta]} \|x(t)\|_{\infty}, \quad (x \in C([\alpha, \alpha + \beta], \mathbb{R}^n)).
\]

Since \( T \) satisfies that for all \( x, y \in C([\alpha, \alpha + \beta], \mathbb{R}^n) \) and for all \( j \geq 1 \),

\[
\|T^j x - T^j y\|_{\infty} \leq \frac{1}{j!} (M \beta)^j \|x - y\|_{\infty}, \quad \text{(2.5)}
\]

with \( M := \max_{\alpha \leq t \leq \alpha + \beta} \|a(t)\|_{\infty} \), then it follows from the Banach Fixed Point Theorem and the convergence of the series \( \sum_{j \geq 1} \frac{(M \beta)^j}{j!} \) for any \( \beta \) and \( M \), that
for each $\bar{x}$ in $C([\alpha, \alpha + \beta], \mathbb{R}^n)$, the sequences $\{T^j\bar{x}\}_{j \geq 1}$ in $C([\alpha, \alpha + \beta], \mathbb{R}^n)$ converges uniformly to the unique solution $u$ of the initial-value problem (1.1) (the fixed point of $T$). Moreover, as a consequence of inequality (2.5) and Taylor’s formula we have the following estimate of the rate of convergence: for all $j \geq 1$,

$$\|T^j\bar{x} - u\|_\infty \leq \frac{(M\beta)^j}{j!} e^{M\beta}\|\bar{T}\bar{x} - \bar{x}\|_\infty. \quad (2.6)$$

### 3. Schauder Bases and Linear Initial-Value Problems

The next result enables to obtain the image under operator $T$ of any continuous function in terms of certain sequences of scalars, sequences which are obtained just by evaluating some functions at adequate points. We shall consider the sup-sup norm on the space $C([\alpha, \alpha + \beta], \mathcal{M}_n(\mathbb{R}))$:

$$\|a\|_\infty := \sup_{t \in [\alpha, \alpha + \beta]} \|a(t)\|_\infty, \quad (a \in C([\alpha, \alpha + \beta], \mathcal{M}_n(\mathbb{R}))).$$

**Theorem 2.** Let $n \geq 1$ and assume that $a = (a_{ij})_{i,j=1,...,n} \in C([\alpha, \alpha + \beta], \mathcal{M}_n(\mathbb{R}))$, $b = (b_j)_{j=1,...,n} \in C([\alpha, \alpha + \beta], \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. Given $1 \leq j, k \leq n$ let $\{a_{jk}^{(i)}\}_{i \geq 1}$ and $\{b_j^{(i)}\}_{i \geq 1}$ be the sequences of scalars satisfying

$$a_{jk} = \sum_{i \geq 1} a_{jk}^{(i)} \Gamma_i \quad \text{and} \quad b_j = \sum_{i \geq 1} b_j^{(i)} \Gamma_i. \quad (3.1)$$

Let us consider the continuous integral operator

$$T : C([\alpha, \alpha + \beta], \mathbb{R}^n) \longrightarrow C([\alpha, \alpha + \beta], \mathbb{R}^n)$$

defined in (2.4). Then, for all $x = (x_j)_{j=1,...,n} \in C([\alpha, \alpha + \beta], \mathbb{R}^n)$ and for all $t \in [\alpha, \alpha + \beta]$ we have that

$$(Tx)(t) = x_0 + \left( \sum_{i \geq 1} c_j^{(i)} \int_\alpha^t \Gamma_i(s)ds \right)_{j=1,...,n},$$

where for $j = 1, \ldots, n$,

$$\begin{aligned}
c_j^{(1)} &= b_j^{(1)} + \sum_{k=1}^n a_{jk}^{(1)} x_k(t_1), \\
c_i^{(i)} &= \sum_{l=1}^i \left( b_j^{(l)} + \sum_{k=1}^n a_{jk}^{(l)} x_k(t_l) \right) \Gamma_k(t_i) - \sum_{l=1}^{i-1} c_j^{(l)} \Gamma_i(t_l), \quad \text{if } i \geq 2. \end{aligned}$$
Proof. Let us start by pointing out that we can assure the existence of the sequences \( \{a_{jk}^i\}_{i \geq 1} \) and \( \{b_{jk}^i\}_{i \geq 1} \), because the functions \( a_{jk} \) and \( b_j \) are continuous. Let us fix a continuous function \( x = (x_j)_{j=1,\ldots,n} \in C([\alpha, \alpha + \beta], \mathbb{R}^n) \). Since for all \( t \in [\alpha, \alpha + \beta] \) we have that
\[
(Tx)'(t) = a(t)x(t) + b(t),
\]
then taking into account (3.1) one arrives at
\[
(Tx)'(t) = \left( b_j(t) + \sum_{k=1}^{n} a_{jk}(t)x_k(t) \right)_j = 1, \ldots, n
\]
\[
= \left( \sum_{i \geq 1} b_j^{(i)}(t) \Gamma_i(t) + \sum_{k=1}^{n} \left( \sum_{i \geq 1} a_{jk}^{(i)} x_k(t) \Gamma_i(t) \right) \right)_j = 1, \ldots, n
\]
\[
= \left( \sum_{i \geq 1} \left( b_j^{(i)} + \sum_{k=1}^{n} a_{jk}^{(i)} x_k(t) \right) \Gamma_i(t) \right)_j = 1, \ldots, n
\]
where the sequence of scalars \( \{c_j^{(i)}\}_{i \geq 1} \), defined as above, is derived from (2.1) and (2.2). Therefore, by integrating the preceding expression one has that
\[
(Tx)(t) = x_0 + \left( \sum_{i \geq 1} c_j^{(i)} \int_{\alpha}^{t} \Gamma_i(s)ds \right)_j = 1, \ldots, n
\]
as required. \( \Box \)

**Theorem 3.** Let \( n \geq 1 \) and assume that \( a = (a_{ij})_{i,j = 1,\ldots,n} \in C([\alpha, \alpha + \beta], \mathcal{M}_n(\mathbb{R})) \), \( b_j = (b_{ij})_{i = 1,\ldots,n} \in C([\alpha, \alpha + \beta], \mathbb{R}^n) \) and \( x_0 \in \mathbb{R}^n \). Let \( T : C([\alpha, \alpha + \beta], \mathbb{R}^n) \to C([\alpha, \alpha + \beta], \mathbb{R}^n) \) be the operator defined in (2.4). Let \( \bar{x} : [\alpha, \alpha + \beta] \to \mathbb{R}^n \) be a continuous function and \( m \geq 1 \) and \( n_1, \ldots, n_m \geq 1 \). Consider the continuous function
\[
y_0(t) := \bar{x}(t) \quad (t \in [\alpha, \alpha + \beta])
\]
and for \( r = 1, \ldots, m \) the continuous functions
\[
L_{r-1}(t) := a(t)y_{r-1}(t) + b(t) \quad (t \in [\alpha, \alpha + \beta]),
\]
and
\[
y_r(t) := x_0 + \int_{\alpha}^{t} (Q_{n_r}(L_{r-1}(s)))_{k=1,\ldots,n}ds \quad (t \in [\alpha, \alpha + \beta]).
\]
Assume in addition that certain positive numbers $\varepsilon_1, \ldots, \varepsilon_m$ satisfy

$$\|T y_{r-1} - y_r\|_{\infty} < \varepsilon_r.$$ 

Then, if $u$ is the solution of the linear initial-value problem (1.1), then we have that

$$\|u - y_m\|_{\infty} \leq \frac{(M \beta)^m}{m!} e^{M \beta} \|T \bar{x} - \bar{x}\|_{\infty} + \sum_{r=1}^{m} \varepsilon_r \frac{(M \beta)^{m-r}}{(m-r)!},$$

where $M = \max_{\alpha \leq t \leq \alpha + \beta} \|a(t)\|_{\infty}$.

**Proof.** Since

$$\|u - y_m\|_{\infty} \leq \|u - T^m \bar{x}\|_{\infty} + \|y_m - T^m \bar{x}\|_{\infty},$$

(3.2)

we shall separately obtain upper bounds for both terms on the left hand side in (3.2). On the one hand, inequality (2.6) gives

$$\|u - T^m \bar{x}\|_{\infty} \leq \frac{(M \beta)^m}{m!} e^{M \beta} \|T \bar{x} - \bar{x}\|_{\infty}.$$ (3.3)

On the other hand, since

for all $v, w \in C([\alpha, \alpha + \beta], \mathbb{R}^n)$ and for all $j \geq 1$,

$$\|T^j v - T^j w\|_{\infty} \leq \frac{1}{j!} (M \beta)^j \|v - w\|_{\infty},$$

then for all $r = 1, \ldots, m$ we have that

$$\|T^{m-r+1} y_{r-1} - T^{m-r} y_r\|_{\infty} = \|T^{m-r} (T y_{r-1}) - T^{m-r} y_r\|_{\infty}$$

$$\leq \frac{1}{(m-r)!} (M \beta)^{m-r} \|T y_{r-1} - y_r\|_{\infty},$$

and hence

$$\|y_m - T^m \bar{x}\|_{\infty} = \|y_m - T^m y_0\|_{\infty} \leq \sum_{r=1}^{m} \|T^{m-r+1} y_{r-1} - T^{m-r} y_r\|_{\infty}$$

$$\leq \sum_{r=1}^{m} \varepsilon_r \frac{(M \beta)^{m-r}}{(m-r)!}. \quad (3.4)$$

Finally, the proof is complete in view of (3.2), (3.3) and (3.4). \qed

Note that given $\varepsilon_1, \ldots, \varepsilon_m > 0$ we can find positive integers $n_1, \ldots, n_m$ such that $\|T y_{r-1} - y_r\|_{\infty} < \varepsilon_r$, since for all $x \in C([\alpha, \alpha + \beta])$, $\lim_{j \geq 1} \|Q_j x - x\|_{\infty} = 0$. 

However, if we wish to find the integers \(m, n_1, \ldots, n_m\) from the positive numbers \(\varepsilon_1, \ldots, \varepsilon_m\), we can use this easy and well-known consequence of the Mean Value Theorem and the interpolating property (2.3) of the basis for \(C([\alpha, \alpha + \beta])\): suppose that \(x \in C^1([\alpha, \alpha + \beta])\) (in fact, we can assume that \(x\) is a continuous and \(C^1\) class function on \([\alpha, \alpha + \beta]\), except perhaps for a finite number of points), \(j \geq 2\) and

\[
h := \max_{i=2, \ldots, j} (s_i - s_{i-1}),
\]

where \(\{s_1 = \alpha < s_2 < \cdots < s_{j-1} < s_j = \alpha + \beta\}\) is the set \(\{t_1, \ldots, t_j\}\) ordered in an increasing way (if \(\|x'\|_\infty = 0\) there is no additional hypothesis on the nodes). Then

\[
\|x - Q_jx\|_\infty \leq 2\|x'\|_\infty h.
\]

If one assumes in the initial-value problem that \(a\) and \(b\) are functions of \(C^1\) class on \([\alpha, \alpha + \beta]\) then the norm appearing in Theorem 3, \(\|T_{y_{r-1}} - y_r\|\) can be estimated as follows:

\[
\|T_{y_{r-1}} - y_r\| \leq \beta\|L_r - (Q_{n_r}(L_r)_{k=1, \ldots, n})\|\]

and then above applies. These ideas are developed more precisely in the following results.

**Lemma 4.** The sequence \(\{L'_r\}_{r \geq 1}\) is uniformly bounded, provided that \(a\) and \(b\) are \(C^1\)-class functions.

**Proof.** It is a well-known fact (Megginson [3]) that the classical Schauder basis for \(C[\alpha, \alpha + \beta]\) is monotone, i.e., for all \(j \geq 1\), \(\|Q_j\| = 1\). Then, for all \(r \geq 1\),

\[
\|y_r(t)\| \leq \|x_0\| + \int_\alpha^t \|L_{r-1}(s)\| ds \leq \|x_0\| + (t - \alpha)(\|a\|\|y_{r-1}\| + \|b\|).
\]

An easy recursive argument gives us that for all \(r \geq 1\) and for all \(t \in [\alpha, \alpha + \beta]\) it holds

\[
\|y_r(t)\| \leq \|x_0\| \sum_{i=0}^{r-1} \|a\|^i(t - \alpha)^i + \|b\| \sum_{i=1}^r \|a\|^{i-1}(t - \alpha)^i + (t - \alpha)^r \|a\|^r \|y_0\|
\]

and so

\[
\|y_r\| \leq (\|x_0\| + \beta\|b\|) \sum_{i=0}^{r-1} (\|a\|\|\beta\|^i + \beta^r \|a\|^r \|y_0\|),
\]

which proves that the sequence \(\{y_r\}_{r \geq 1}\) is uniformly bounded. On the other hand, since

\[
y'_r(t) = Q_{n_r}(a(t)y_{r-1}(t) + b(t)),
\]
then we arrive at the fact that the sequence \( \{y'_r\}_{r \geq 1} \) also is bounded, because of the uniform boundness of \( \{y_r\}_{r \geq 1} \) and the monotonicity of the classical Schauder basis. Finally, the announced statement follows from the uniform boundness of the sequences \( \{y_r\}_{r \geq 1} \) and \( \{y'_r\}_{r \geq 1} \) and the equality

\[
L'_r(t) = a'(t)y_{r-1} + a(t)y'_{r-1} + b'(t).
\]

\[\square\]

**Theorem 5.** Assume that \( a = (a_{ij})_{ij=1,...,n} \in C^1([\alpha, \alpha + \beta], \mathcal{M}_n(\mathbb{R})) \) and \( b = (b_j)_{j=1,...,n} \in C^1([\alpha, \alpha + \beta], \mathbb{R}^n) \). Then there exists \( K > 0 \) such that for all \( m \geq 1 \) we have that

\[
\sum_{r=1}^{m} \|Ty_{r-1} - y_r\|_\infty \frac{(M\beta)^{m-r}}{(m-r)!} \leq Kh.
\]

**Proof.** In view of (3.5), we have that for all \( r \geq 1 \)

\[
\|Ty_{r-1} - y_r\|_\infty \leq \beta \|L_r - (Q_n, (L_r)_k)_{k=1,...,n}\|_\infty \leq 2\beta \|L'_r\|_\infty h
\]

and since the sequence \( \{L'_r\}_{r \geq 1} \) is uniformly bounded (Lemma 4), then there exists \( K_1 > 0 \) such that for all \( r \geq 1 \) we have that

\[
\|Ty_{r-1} - y_r\|_\infty \leq K_1 h.
\]

Therefore, we deduce that

\[
\sum_{r=1}^{m} \|Ty_{r-1} - y_r\|_\infty \frac{(M\beta)^{m-r}}{(m-r)!} \leq K_1 he^{M\beta}.
\]

Take \( K := K_1 e^{M\beta} \) and the proof is complete. \[\square\]

**Remark 6.** Although our numerical method works for any Schauder basis in the Banach space \( C([\alpha, \alpha + \beta]) \), we have chosen the classical one because the biorthogonal functionals and the projections associated have an easy expression. Let us also point out that Schauder bases have been used as a fundamental tool in order to solve numerically some integral equations (Berenguer et al [1]).
4. A Numerical Example

Finally we exhibit an example which shows the behaviour of our results. To this end, we fix the data’s initial-value problem: $x_0 \in \mathbb{R}^n$, $a = (a_{ij})_{i,j=1,...,n} \in C^1([\alpha, \alpha + \beta], M_n(\mathbb{R}))$ and $b = (b_j)_{j=1,...,n} \in C^1([\alpha, \alpha + \beta], \mathbb{R}^n)$ and take $\bar{x} = x_0$.

We choose an $n \in \mathbb{N}$ with $n = 2^k + 1$, $k \in \mathbb{N}$, and thus

$$h = \max_{2 \leq i \leq n} (s_i - s_{i-1}) = \frac{1}{2^k}.$$ 

Then we calculate the sequences of coefficients $\{a_{ij}^{(i)}\}_{i=1}^n$ and $\{b_j^{(i)}\}_{i=1}^n$ and obtain recursively the functions $y_r$ in Theorem 3, taking $n_1 = \cdots = n_r = n$ and $\bar{x} = x_0$.

With such a choose for $\bar{x}$ we have that

$$T\bar{x}(t) - \bar{x}(t) = \int_{\alpha}^{t} (a(s)\bar{x}(s) + b(s))ds$$

and the norm $\|T\bar{x} - \bar{x}\|_{\infty}$ in Theorem 3 can be easily bounded. We also determine the errors

$$E_{nr} = \max_i |y_r(s_i) - u(s_i)|,$$

where $u$ is the exact solution. We have considered the approximation of the exact solution $y_m$ in such a way that

$$\left| \frac{E_{nm}}{E_{nm+1}} \right| < 1 + 10^{-2}.$$

**Example.** Consider the initial-value problem

$$\begin{cases}
x'(t) = \begin{pmatrix} \frac{1}{t+2} & 0 \\ \frac{3t+1}{\sqrt{t}} \end{pmatrix} x(t) + \begin{pmatrix} -\sin t + \cos t \\ -\sin t + \sqrt{t}\cos t - (3t+1)\sin t \end{pmatrix}, \\
x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{cases}$$

whose exact solution is $u(t) = (\sin t, \cos t)$. In the columns of the following table we give the absolute errors $E_{nm}$ in nine representative points of the approximations $y_m$, obtained with different values of $n$. 

\begin{array}{|c|c|c|}
\hline
& (n = 9, m = 7) & (n = 17, m = 8) & (n = 33, m = 8) \\
\hline
0 & (0, 0) & (0, 0) & (0, 0) \\
0.125 & (1.67 \times 10^{-4}, 4.31 \times 10^{-6}) & (4.18 \times 10^{-5}, 8 \times 10^{-7}) & (1.04 \times 10^{-5}, 1.82 \times 10^{-7}) \\
0.250 & (3.41 \times 10^{-4}, 2.66 \times 10^{-5}) & (8.53 \times 10^{-6}, 6.05 \times 10^{-6}) & (2.13 \times 10^{-5}, 1.47 \times 10^{-6}) \\
0.375 & (5.19 \times 10^{-5}, 8.71 \times 10^{-6}) & (1.29 \times 10^{-5}, 2.07 \times 10^{-6}) & (3.24 \times 10^{-6}, 5.13 \times 10^{-6}) \\
0.500 & (6.98 \times 10^{-5}, 2.08 \times 10^{-4}) & (1.74 \times 10^{-5}, 5.07 \times 10^{-6}) & (4.36 \times 10^{-6}, 1.25 \times 10^{-5}) \\
0.625 & (8.73 \times 10^{-5}, 4.18 \times 10^{-4}) & (2.18 \times 10^{-4}, 1.02 \times 10^{-4}) & (5.46 \times 10^{-5}, 2.54 \times 10^{-5}) \\
0.750 & (1.04 \times 10^{-4}, 7.48 \times 10^{-5}) & (2.61 \times 10^{-4}, 1.83 \times 10^{-4}) & (6.52 \times 10^{-5}, 4.57 \times 10^{-5}) \\
0.875 & (1.20 \times 10^{-3}, 1.23 \times 10^{-3}) & (3.01 \times 10^{-4}, 3.02 \times 10^{-4}) & (7.53 \times 10^{-5}, 7.58 \times 10^{-5}) \\
1 & (1.35 \times 10^{-3}, 1.94 \times 10^{-3}) & (3.39 \times 10^{-4}, 4.71 \times 10^{-4}) & (8.48 \times 10^{-5}, 1.19 \times 10^{-4}) \\
\hline
\end{array}

References


