LOWER BOUNDS FOR THE SUM DIVISOR FUNCTION

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Abstract: Let $\sigma(n)$ be the sum divisor function and let $s(n)$ denote the square-free kernel of positive integer $n$. We prove that for every $k$, such that $2 \leq k \leq r = \omega(n)$ we have (*) $\sigma(n) > (\sqrt[n] {n} + \sqrt[n] {n_0})^r \geq (\sqrt[n] {n} + \sqrt[n] {n_0})^k$, where $n_0 = \frac{n}{s(n)}$ and $\omega(n)$ is the number of distinct prime divisor of $n$. Moreover, we prove that for infinitely many $n$, we have $\sigma(n) > \frac{6}{\pi^2} e^\gamma n \log \log n$, where $\gamma \approx 0.57721$ is Euler’s constant.

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1. Introduction

The purpose of this paper is to prove some lower bounds for the sum divisor function $\sigma(n) = \sum_{d|n} d$. Namely, we prove of the following theorem.

**Theorem 1.** Let $n$ be a composite positive integer and let $r = \omega(n)$ be the number of all distinct prime divisor of $n$. Moreover, let $s(n)$ denote the square-free kernel of $n$. Then for every $k$ such that $2 \leq k \leq r$ we have

$$\sigma(n) > (\sqrt[n] {n} + \sqrt[n] {n_0})^r \geq (\sqrt[n] {n} + \sqrt[n] {n_0})^k,$$

where $n_0 = \frac{n}{s(n)}$. 
Immediately from Theorem 1 it follows the following corollary.

**Corollary 1.** If $r = \omega(n) \geq 2$ then

$$\sigma(n) > (\sqrt{n} + 1)^r \geq n + 2\sqrt{n} + 1. \quad (***)$$

We note that the inequality (***) is better than classical inequality $\sigma(n) > n + \sqrt{n}$ presented in the Sierpiński's monograph [7], on page 180.

Further, we consider the function $\gamma_n : [1, \infty) \to \mathbb{R}_+$, defined by the rule:

$$\frac{1}{\gamma_n(x)} = x \cdot \left( \left( \frac{\sigma(n)}{n} \right)^{1/x} - 1 \right). \quad (1.1)$$

It is easy to observe that the first derivative of the function (1.1) is negative and moreover the function $\gamma_n$ increases to $\frac{1}{\log \frac{\sigma(n)}{n}}$ with $t \to \infty$. Hence by the Theorem 1 it follows the following corollary.

**Corollary 2.** If $n$ is a composite positive integer, then

$$s(n) > (\gamma_n(r) \cdot r)^r \geq \left( \frac{n}{\sigma(n) - n} \cdot r \right)^r,$$

where $s(n)$ is the square-free kernel of $n$ and $r = \omega(n)$.

The proof of the Theorem 1 is based on a special version of the Minkowski inequality (see [1], Chapter 2)

$$\prod_{i=1}^{k}(1 + x_i) \geq (1 + \sqrt[k]{x_1 \cdots x_k})^k, \quad (M)$$

where $x_j \geq 0$ for $j = 1, 2, \ldots, k$.

We also note that the inequality (M) has been used by A. Grytczuk and M. Wójtowicz [3] in the proof of an upper bound for the Euler’s totient function. We prove also in this paper the following theorem.

**Theorem 2.** Let $\sigma(n)$ be the sum divisor function. Then for infinitely many positive integers $n$, we have

$$\sigma(n) > \frac{6}{\pi^2} e^{\gamma n} \log \log n. \quad (***)$$

We note that such type estimation as in (***) is strictly connected with the Riemann Hypothesis. The Riemann zeta function $\zeta(s)$ for $s = \sigma + it$ is defined by Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$
which converges for $\sigma > 1$ and it has analytic continuation to the complex plane with one singularity as a simple pole and $\text{res}\zeta(s) = 1$.

In 1859 Riemann [5] conjectured that the nonreal zeros of the Riemann zeta function $\zeta(s)$ all lie on the line $\sigma = \frac{1}{2}$.

The connection of the Riemann Hypothesis with prime numbers has been considered by Gauss.

Let $\pi(x) = \sum_{p \leq x} 1$, then it is well-known that the Riemann Hypothesis is equivalent to the assertion that for each $\varepsilon > 0$ there is a positive constant $c = c(\varepsilon)$ such that $|\pi(x) - \text{Li}(x)| \leq c(\varepsilon)x^{\frac{1}{2}+\varepsilon}$, where $\text{Li}(x) = \int_{2}^{x} \frac{dt}{\log t}$.

Many others equivalent results with the Riemann Hypothesis are known. We note that more of them has been presented by Conrey in very nice article [2].

We concern only to one criterion, but very interesting criterion given by Robin [6] in 1984. Robin proved that the Riemann Hypothesis is true if and only if

$$\sigma(n) < e^\gamma n \log \log n, \quad (R)$$

for all positive integers $n \geq 5041$, where $\gamma \approx 0.57721$ is the Euler’s constant.

### 2. Proof of Theorem 1

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ then we have

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{1+\alpha_i} - 1}{p_i - 1} = \prod_{i=1}^{r} \left( p_i^{\alpha_i} + p_i^{\alpha_i-1} + \ldots + p_i + 1 \right). \quad (2.1)$$

From (2.1) we have

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i} + \ldots + \frac{1}{p_i^{\alpha_i}} \right). \quad (2.2)$$

By (2.2) it follows that

$$\frac{\sigma(n)}{n} > \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i} \right). \quad (2.3)$$
Putting \( x_i = \frac{1}{p_i} \) in the Minkowski inequality (M) from (2.3) we obtain

\[
\frac{\sigma(n)}{n} > \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i} \right) \geq \left( 1 + \frac{1}{\sqrt[r]{\prod_{i=1}^{r} p_i}} \right)^r.
\]  

(2.4)

Since \( s(n) = \prod_{i=1}^{r} p_i \) and \( n_0 = \frac{n}{s(n)} \) then by (2.4) it follows that

\[
\sigma(n) > (\sqrt[n]{n} + \sqrt[n]{n_0})^r,
\]

(2.5)

and we see that (2.5) proves the first inequality in (*) of Theorem 1.

For the proof of the second part of the inequality in (*) we consider the function: \( t \to \left( 1 + \xi \cdot a^t \right)^t \) defined on \([1, +\infty)\) for \( a \in (0, 1)\).

It is easy to observe that this function is increasing for \( \xi = 1 \). From this fact follows the second inequality in (*) and the proof of Theorem 1 is complete.

3. Proof of Theorem 2.

For the proof of Theorem 2 we consider the following expression:

\[
\frac{\sigma(n)\varphi(n)}{n^2},
\]

(3.1)

where \( \sigma(n) = \sum_{d|n} d \), and \( \varphi(n) \) is Euler’s totient function. Let \( n = \prod_{i=1}^{r} p_i^{\alpha_i} \). Then we have

\[
\sigma(n) = \prod_{i=1}^{r} p_i^{1+\alpha_i} - 1 = \frac{n \cdot \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^{1+\alpha_i}} \right)}{\prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right)},
\]

(3.2)

\[
\varphi(n) = n \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right).
\]

(3.3)

From (3.2) and (3.3) we obtain

\[
\frac{\sigma(n)\varphi(n)}{n^2} = \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^{1+\alpha_i}} \right).
\]

(3.4)
By (3.4) it follows that
\[
\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{1+\alpha_i}}\right). \tag{3.5}
\]
Since \(\alpha_i \geq 1\) for \(i = 1, 2, \ldots, r\) then we have
\[
\prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{1+\alpha_i}}\right) \geq \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{\alpha_i}}\right). \tag{3.6}
\]
Let \(P\) denote the set of all primes, then it is easy to see that
\[
\prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{\alpha_i}}\right) > \prod_{p \in P} \left(1 - \frac{1}{p^2}\right). \tag{3.7}
\]
On the other hand it is well-known that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \prod_{p \in P} \left(1 - \frac{1}{p^2}\right)^{-1}. \tag{3.8}
\]
By (3.6)-(3.8) and (3.5) it follows that
\[
\frac{\sigma(n)}{n} > \frac{6}{\pi^2} \frac{n}{\varphi(n)}. \tag{3.9}
\]
Applying to (3.9) the result given by Nicolas [4] that for infinitely many natural \(n\) we have
\[
\frac{n}{\varphi(n)} > e^\gamma \log \log n,
\]
we obtain
\[
\sigma(n) > \frac{6}{\pi^2} e^\gamma n \log \log n,
\]
and the proof of Theorem 2 is complete.

References


