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DIFFERENTIAL STRUCTURES OF RIEMANNIAN HOMOGENEOUS SPACE $R^{2n}\backslash(0)$

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Abstract: The purpose of this paper is to construct a $2n$-dimensional Riemannian manifold (M, g) of which the hypersurface S^{2n-1} is totally geodesic and to study differential structures of such an ambient manifold (M, g).

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1. Introduction and Main Results

In contrast to general Riemannian manifolds, globally symmetric spaces contain plenty of totally geodesic submanifolds. The following theorem is well known.

Theorem A. (cf. [4, p. 224, Theorem 7.2]) *Let G/H be a Riemannian globally symmetric space with the canonical decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Let $\mathfrak{s}$ be a Lie triple system contained in $\mathfrak{m}$. Put $S = \mathrm{Exp}\,\mathfrak{s}$. Then, S has a natural differentiable structure in which it is a totally geodesic submanifold of G/H.*

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In this paper, from a new point of view, for a given Riemannian manifold (S, g) we construct an ambient manifold (M, g) so that (S, g) may be a totally geodesic submanifold, and then study differential structures of the ambient manifold (M, g). A $(n-1)$ dimensional Euclidean unit sphere S^{n-1} is not a geodesic hypersurface of Euclidean space E^n. It is interesting to ask the following:

May you construct an n*-dimensional Riemannian manifold* (M, g) *of which the hypersurface* S^{n-1} *is totally geodesic?*

In this paper, we construct such ambient Riemannian manifolds (M, g) as in the above question, and then study differential structures of such ambient manifolds (M, g).

The following lemma is very useful in our paper.

Lemma B. (cf. [8]) *Let* g *be a left invariant Riemannian metric on* $SU(2)$. *Let* $<\ ,\ >$ *be an inner product on* $\mathfrak{su}(2)$ *defined by* $< X, Y >:= g_e(X_e, Y_e)$, *where* $\mathfrak{su}(2)$ *is the Lie algebra of* $SU(2)$, $X, Y \in \mathfrak{su}(2)$, *and* e *is the identity matrix of* $SU(2)$. *Then there exists an orthonormal basis* (V_1, V_2, V_3) *of* $\mathfrak{su}(2)$ *with respect to* $<\ ,\ >_o$ *such that*

$$\begin{cases} [V_1,\ V_2] \ = \ (1/\sqrt{2})V_3, \qquad [V_2,\ V_3] \ = \ (1/\sqrt{2})V_1, \\ [V_3,\ V_1] \ = \ (1/\sqrt{2})V_2, \qquad < V_i,\ V_j > = \ \delta_{ij}\ a_i^2, \end{cases} \tag{1.1}$$

where $<\ ,\ >_o$ *is an inner product induced from the Killing form of* $\mathfrak{su}(2)$ *and* a_i, $(i = 1, 2, 3)$, *are positive constant real numbers determined by the given left invariant Riemannian metric* g *of* $SU(2)$.

For a given left invariant Riemannian metric g on $SU(2)$, we fix an orthonormal basis $(V_1,\ V_2,\ V_3)$ of $\mathfrak{su}(2)$ with respect to $<\ ,\ >_o$ with the property (1.1) in Lemma B and denote by $g(a_1, a_2, a_3)$, or simply by $g(a)$, the left invariant Riemannian metric on $SU(2)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma B.

Retaining the same notations as in Lemma B, we states the following results which we got in this paper.

Theorem 1. *Let* $g(a)$ *be a left invariant Riemannian metric on* $SU(2)$. *Let* g *be a left invariant Riemannian metric on* $R^4\backslash(0)$ *determined by an orthonormal basis* $\{X_1 := V_1/a_1,\ X_2 := V_2/a_2,\ X_3 := V_3/a_3,\ X_4 := I_2 + \sum_{i=1}^{3} c_i(V_i/a_i)\}$ *on* $T_e(R^4\backslash(0))$, *where each* c_i *is a real constant. Let* J *be a tensor field of type* $(1, 1)$ *on* $R^4\backslash(0)$ *defined by* $JX_1 = X_4,\ JX_2 = X_3,\ JX_3 = -X_2,\ JX_4 = -X_1$.

Then, a necessary and sufficient condition in order for the structure tensor field J of the almost complex manifold $(R^4\backslash(0), g, J)$ to be a complex structure is $a_2 = a_3$ and $c_2 = c_3 = 0$.

Moreover, if the structure tensor field J of the ambient manifold $R^4\backslash(0), g, J$ is a complex structure, then its hypersurface $(S^3, g(a))$ is totally geodesic.

Theorem 2. *A necessary and sufficient condition for the Riemannian manifold $(R^4\backslash(0), g)$ in Theorem 1 to be naturally reductive is $a_1 = a_2 = a_3$ and $c_1 = c_2 = c_3 = 0$.*

Corollary 3. *If the Riemannian manifold $(R^4\backslash(0), g)$ in Theorem 1 is a naturally reductive Riemannian homogeneous space, its hypersurface S^3 is totally geodesic.*

Note. The complete classification of the simply connected four dimensional naturally reductive homogeneous spaces is given in [3].

From Theorem 1, we can get almost complex manifolds which are not complex manifolds. And, it is shown (cf. Remark 1) that any almost complex manifold $(R^4\backslash(0), g, J)$ in Theorem 1 is not an almost Kaehler manifold.

In this paper, we use the following notations:

$R^\times := R\backslash(0)$; I_n: the unit matrix of order n;

$G := R^\times SU(n) = \{ rg \mid r \in R^\times,\ g \in SU(n) \}$,

E_{ij}: a square matrix of order n with entry 1 where the i-th row and j-th column meet, all other entries being 0.

Then, we obtain the following results.

Theorem 4. *Let $\tilde{g}$ be a $SU(n)$-invariant Riemannian metric on S^{2n-1} $(\cong SU(n)/SU(n-1))$ which is determined by an orthonormal basis*

$$\{X_j, Y_j\}_{j=2}^{n} \bigcup \{Y_1 := \sqrt{-2/(n^2-n)}((1-n)E_{11} + \sum_{i=2}^{n} E_{ii})\}$$

on $T_o(S^{2n-1})$, where $o := \{SU(n-1)\}$, $X_j := E_{1j} - E_{j1}$, $Y_j := \sqrt{-1}(E_{1j} + E_{j1})$. Let g be the left G-invariant Riemann metric on $R^{2n}\backslash(0)$ which is determined by an orthonormal basis

$$\{X_j, Y_j\}_{j=2}^{n} \bigcup \{X_1 := I_n,\ Y_1\}$$

on $T_o(R^{2n}\backslash(0))$. Let $\tilde{J}$ be a tensor of type $(1,1)$ on $T_o(R^{2n}\backslash(0))$ defined by $\tilde{J}X_b = Y_b$, $\tilde{J}Y_b = -X_b$, $(b = 1, 2, ..., n)$.

Then there exist a tensor field J of type $(1,1)$ on $R^{2n}\backslash(0)$ such that $J_o = \tilde{J}$, and $(R^{2n}\backslash(0), g, J)$ is a complex manifold of which the hypersurface S^{2n-1} is totally geodesic.

Theorem 5. *Every Riemannian manifold* $(R^{2n}\backslash(0), g)$, $n \geq 3$, *in Theorem* 4 *is a naturally reductive Riemannian homogeneous space which is not symmetric.*

It is shown that any almost complex manifold$(R^{2n}\backslash(0), g, J)$ in Theorem 4 is not almost Kaehler (cf. Remark 4).

2. Preliminaries

A homogeneous space K/T of a connected Lie group K is called *reductive* if the following condition is satisfied: in the Lie algebra $\mathfrak{k}$ of K there exists a subspace $\mathfrak{m}$ such that $\mathfrak{k} = \mathfrak{m} + \mathfrak{t}$ (direct sum of vector spaces) and $\mathrm{Ad}(t)\mathfrak{m} \subset \mathfrak{m}$ for all $t \in T$, where $\mathfrak{t}$ is the subalgebra of $\mathfrak{k}$ corresponding to the identity component T_o of T and $\mathrm{Ad}(t)$ denotes the adjoint representation of T in $\mathfrak{k}$.

Let $< , >$ be an inner product which is invariant with respect to $\mathrm{Ad}(T)$ on $\mathfrak{m}$. This inner product $< , >$ determines an invariant Riemannian metric g on K/T. Then the connection function α (cf. [7, p. 43]) on $\mathfrak{m}\times\mathfrak{m}$ corresponding to the invariant Riemannian connection of a reductive Riemannian homogeneous space $(K/T, g)$ is given as follows (cf. [7, p. 52]):

$$\alpha(X,Y) \;=\; (1/2)\;[X,Y]_{\mathfrak{m}} \;+\; U(X,Y) \qquad (X,Y \in \mathfrak{m}), \tag{2.1}$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > = < [Z,X]_{\mathfrak{m}}, Y > + < X, [Z,Y]_{\mathfrak{m}} > \qquad (X,Y,Z \in \mathfrak{m}), \tag{2.2}$$

where $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element of $X \in \mathfrak{k} = \mathfrak{t}+\mathfrak{m}$. A reductive Riemannian homogeneous space $(K/T, g)$ with the condition $U = 0$ on $\mathfrak{m}\times\mathfrak{m}$ is said to be a *natually reductive Riemannian homogeneous space.*

Let a Riemannian manifold (M^{2n}, g) have a tensor field J of type $(1,1)$ such that $J^2 = -I$ and $g(JX, JY) = g(X, Y),\ X, Y \in \mathfrak{X}(M)$. Such a manifold (M^{2n}, g, J) is said to be an *almost complex manifold*. A tensor field N of type $(1,2)$ on an almost complex manifold (M^{2n}, g, J) given by

$$N(X,Y) := J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY], \tag{2.3}$$

$X, Y \in \mathfrak{X}(M^{2n})$, is said to be *Nijenhuis tensor field* (cf. [6, p. 10]). A necessary and sufficient condition for an almost complex manifold (M^{2n}, g, J) to be an n-dimensional complex manifold is $N = 0$ on $\mathfrak{X}(M)\times\mathfrak{X}(M)$ (cf. [5, p. 124]).

The *Kaehler form* ω on (M^{2n}, g, J) is defined by $\omega(X,Y) := g(X, JY)$. An almost complex manifold (M, g, J) is called *almost Kaehler* if the Kaehler form ω is closed, that is, $d\omega = 0$. A complex manifold with $d\omega = 0$ is called *Kaehler*.

3. Proofs of Main Theorems

3.1. Proofs of Theorem 1, Theorem 2 and Corollary 3

We retain the notations as in Section 1 and Section 2. We identify $R^4\backslash(0)$ with $R^{\times}SU(2)$. We put $\mathfrak{m} := T_e(G)$ and $r := \sqrt{2}a_1a_2a_3$. Here a_1, a_2, a_3 are constants appearing in $(S^3, g(a))$. We get from (1.1)

$$\begin{aligned} &[X_1, X_2] = ({a_3}^2 X_3)/r, && [X_1, X_3] = (-{a_2}^2 X_2)/r,\\ &[X_1, X_4] = (-{a_2}^2 c_3 X_2 + {a_3}^2 c_2 X_3)/r, && [X_2, X_3] = ({a_1}^2 X_1)/r,\\ &[X_2, X_4] = ({a_1}^2 c_3 X_1 - {a_3}^2 c_1 X_3)/r, &&\\ &[X_3, X_4] = (-{a_1}^2 c_2 X_1 + {a_2}^2 c_1 X_2)/r. && \end{aligned} \tag{3.1}$$

Using (2.3) and (3.1), we have

$$\begin{aligned} N(X_1, X_2) &= \{{a_1}^2 c_3 X_1 + ({a_2}^2 - {a_3}^2) X_2\\ &\qquad + ({a_2}^2 - {a_3}^2) c_1 X_3 - {a_1}^2 c_2 X_4\}/r,\\ N(X_1, X_3) &= \{-{a_1}^2 c_2 X_1 + ({a_2}^2 - {a_3}^2) c_1 X_2\\ &\qquad + (-{a_2}^2 + {a_3}^2) X_3 - {a_1}^2 c_3 X_4\}/r,\\ N(X_1, X_4) &= N(X_2, X_3) = 0,\\ N(X_2, X_4) &= \{{a_1}^2 c_2 X_1 + (-{a_2}^2 + {a_3}^2) c_1 X_2\\ &\qquad + ({a_2}^2 - {a_3}^2) X_3 + {a_1}^2 c_3 X_4\}/r,\\ N(X_3, X_4) &= \{{a_1}^2 c_3 X_1 + ({a_2}^2 - {a_3}^2) X_2\\ &\qquad + ({a_2}^2 - {a_3}^2) c_1 X_3 - {a_1}^2 c_2 X_4\}/r. \end{aligned} \tag{3.2}$$

From (3.2), we obtain the fact that a necessary and sufficient condition for the almost complex manifold $(R^4\backslash(0), g, J)$ determined by the conditions in Theorem 1 to be a complex manifold is

$$a_2 = a_3 \quad \text{and} \quad c_2 = c_3 = 0. \tag{3.3}$$

On the other hand, we have from (2.2)

$$\begin{aligned}
U(X_1,X_2) &= \{(-{a_1}^2+{a_2}^2)X_3+(-{a_1}^2+{a_2}^2)c_3X_4\}/2r,\\
U(X_1,X_3) &= \{({a_1}^2-{a_3}^2)X_2+({a_1}^2-{a_3}^2)c_2X_4\}/2r,\\
U(X_1,X_4) &= \{{a_1}^2c_3X_2-{a_1}^2c_2X_3\}/2r,\\
U(X_2,X_3) &= \{(-{a_2}^2+{a_3}^2)X_1+(-{a_2}^2+{a_3}^2)c_1X_4\}/2r,\\
U(X_2,X_4) &= (-{a_2}^2c_3X_1+{a_2}^2c_1X_3)/2r,\\
U(X_3,X_4) &= ({a_3}^2c_2X_1-{a_3}^2c_1X_2)/2r,\\
U(X_1,X_1) &= U(X_2,X_2)=U(X_3,X_3)=U(X_4,X_4)=0.
\end{aligned} \tag{3.4}$$

We have from (2.1), (3.1) and (3.3)

$$\begin{aligned}
\alpha(X_1,X_4) &= \{({a_1}^2-{a_2}^2)c_3X_2+(-{a_1}^2+{a_3}^2)c_2X_3\}/2r,\\
\alpha(X_2,X_4) &= \{({a_1}^2-{a_2}^2)c_3X_1+({a_2}^2-{a_3}^2)c_1X_3\}/2r,\\
\alpha(X_3,X_4) &= \{(-{a_1}^2+{a_3}^2)c_2X_1+({a_2}^2-{a_3}^2)c_1X_2\}/2r.
\end{aligned} \tag{3.5}$$

From (3.3) and (3.5), we deduce that if the almost complex manifold $(R^4\backslash(0),g,J)$ determined by the conditions in Theorem 1 is is a complex manifold, then its hypersurface $(S^3,g(a))$ is totally geodesic.

Thus, Theorem 1 is obtained.

From (3.4), we can get Theorem 2. Moreover, combining (3.5) with Theorem 2, we obtain Corollary 3.

Remark 1. Since $d\omega(X_2,X_3,X_4)=-{a_1}^2/r$ is not zero, $(R^4\backslash(0),g,J)$ in Theorem 1 is not an almost Kaehler manifold.

Remark 2. From (2.1),(3.1) and (3.4), we have

$$\alpha(X_1,X_2)=\{(-{a_1}^2+{a_2}^2+{a_3}^2)X_3+c_3(-{a_1}^2+{a_2}^2)X_4\}/2r\,,$$

$$\alpha(X_2,X_1)=\{(-{a_1}^2+{a_2}^2-{a_3}^2)X_3+c_3(-{a_1}^2+{a_2}^2)X_4\}/2r.$$

If the Riemannian manifold $(R^4\backslash(0),g)$ in Theorem 1 is symmetric, $\alpha(X_1,X_2)=\alpha(X_2,X_1)=0$, whence $a_3=0$, which is an absurd since $a_3>0$. Thus, the Riemannian manifold $(R^4\backslash(0),g)$ in Theorem 1 having $(S^3,g(a))$ as its hypersurface is not symmetric.

Remark 3. $S^1=\{\mathrm{diag}\,[e^{\sqrt{-1}\theta},e^{-\sqrt{-1}\theta}]|\ \theta\in R\}\subsetneqq R^2\backslash(0)=\{rB|\ r\in R^\times,\ B\in S^1\}$. Here, $\mathrm{diag}\,[x,y]$ denotes a diagonal matrix of order 2 whose diagonal entries are x and y. By the similar way as in Theorem 1, we can construct the Riemannian structure in $R^2\backslash(0)$ so that its hypersurface S^1 may be totally geodesic.

3.2. Proofs of Theorem 4 and Theorem 5

We retain the notations as in Section 1 and Section 2. We identify $R^{2n}\backslash(0)$ (resp. S^{2n-1}) with $G/SU(n-1)$ (resp. $SU(n)/SU(n-1)$). First of all we define an $\mathrm{Ad}(SU(n-1))$-invariant inner product $< \ , \ >$ on the Lie algebra $\mathfrak{g} = \mathfrak{su}(n) + RI_n$ of G by

$$< X, Y > = \begin{cases} (-1/2)\ \mathrm{trace}\,(XY), & X, Y \in \mathfrak{su}(n), \\ (-1/2)\ \mathrm{trace}\,(XY), & X \in \mathfrak{su}(n), \quad Y \in RI_n, \\ (1/n)\ \mathrm{trace}\,(XY), & X, Y \in RI_n. \end{cases} \tag{3.6}$$

We define by $\mathfrak{m}$ (resp. $\mathfrak{m}'$) the subspace of $\mathfrak{g}$ (resp. $\mathfrak{su}(n)$) which is orthogonal to $\mathfrak{su}(n-1)$ with respect to $< \ , \ >$. Then, $\{X_i, Y_i\}_{i=1}^n$ (resp. $\{X_i, Y_i\}_{i=2}^n \cup \{Y_1\}$) in Theorem 4 is an orthonormal basis of $\mathfrak{m}$ (resp. $\mathfrak{m}'$) with respect to $< \ , \ >$. Then we have

$$\begin{aligned} &[X_1, Y_1\,] = [X_1, X_k] = [X_1, Y_k] = 0, \\ &[Y_1\ , X_k] = -\sqrt{2n/(n-1)}Y_k, \quad [Y_1, Y_k] = \sqrt{2n/(n-1)}X_k, \\ &[X_k, X_l\,] = -(E_{kl} - E_{lk}) \in \mathfrak{su}(n-1), \ [X_k, Y_k] = -\sqrt{2n/(n-1)}Y_1, \\ &[X_k, Y_l\ ] = -(E_{kl} + E_{lk}) \in \mathfrak{su}(n-1), \\ &[Y_k\ , Y_l\ ] = -(E_{kl} - E_{lk}) \in \mathfrak{su}(n-1). \end{aligned} \tag{3.7}$$

Here and from now on, j, k, l run over $2,\ 3,\ 4, \cdots,\ n,\ \ (k \neq l)$, and $a,\ b$ run over $1,\ 2,\ 3, \cdots,\ n$.

Then there exists a tensor field J of type (1,1) on $R^{2n}\backslash(0)$ such that $J^2 = -I$ and $J_o = \tilde{J}$ (cf. [9]) if and only if

$$(\mathrm{ad}\,(X) \circ \tilde{J})(Y) \ = \ (\tilde{J} \circ \mathrm{ad}\,(X))(Y) \tag{3.8}$$

for all $X \in \mathfrak{su}(n-1)$ and $Y \in \mathfrak{m}$. A basis of $\mathfrak{su}(n-1)$ is given by

$$\bigcup_{k,l} \{E_{kl} - E_{lk},\ \sqrt{-1}(E_{kl} + E_{lk}),\ \sqrt{-1}(E_{kk} - E_{k+1k+1})\}. \tag{3.9}$$

Then we have

$$
\begin{aligned}
&[E_{kl} - E_{lk}, X_j] \;=\; \delta_{jl}X_k - \delta_{jk}X_l,\\
&[\sqrt{-1}(E_{kl} + E_{lk}), X_j] \;=\; -\delta_{jl}Y_k - \delta_{jk}Y_l,\\
&[\sqrt{-1}(E_{kk} - E_{k+1k+1}), X_j] \;= -\delta_{kj}Y_k + \delta_{jk+1}Y_{k+1},\\
&[E_{kl} - E_{lk}, Y_j] \;=\; \delta_{lj}Y_k - \delta_{kj}Y_l,\\
&[\sqrt{-1}(E_{kl} + E_{lk}), Y_j] \;=\; \delta_{lj}X_k + \delta_{kj}X_l,\\
&[\sqrt{-1}(E_{kk} - E_{k+1k+1}), Y_j] \;=\; \delta_{jk}X_k - \delta_{jk+1}X_{k+1},\\
&[E_{kl} - E_{lk}, X_1] \;=\; [E_{kl} - E_{lk}, Y_1] \;=\; 0,\\
&[\sqrt{-1}(E_{kl} + E_{lk}), X_1] \;=\; [\sqrt{-1}(E_{kl} + E_{lk}), Y_1] \;= 0,\\
&[\sqrt{-1}(E_{kk} - E_{k+1k+1}), X_1] \;=\; [\sqrt{-1}(E_{kk} - E_{k+1k+1}), Y_1] = \;0.
\end{aligned}
\tag{3.10}
$$

Moreover, we get from (3.10)

$$
\begin{aligned}
&\tilde{J}(\mathrm{ad}\,(E_{kl} - E_{lk})X_j) = \mathrm{ad}\,(E_{kl} - E_{lk})\tilde{J}(X_j) = \delta_{jl}Y_k - \delta_{jk}Y_l,\\
&\tilde{J}(\sqrt{-1}\mathrm{ad}\,(E_{kl} + E_{lk})X_j) = \sqrt{-1}\mathrm{ad}\,(E_{kl} - E_{lk})\tilde{J}(X_j)\\
&\qquad\qquad = \delta_{jl}X_k + \delta_{jk}X_l\;,\\
&\tilde{J}(\sqrt{-1}\mathrm{ad}\,(E_{kk} - E_{k+1k+1})X_j) = \sqrt{-1}\mathrm{ad}\,(E_{kk} - E_{k+1k+1})\tilde{J}(X_j)\\
&\qquad\qquad = \delta_{jk}X_k - \delta_{jk+1}X_{k+1},\\
&\tilde{J}(\mathrm{ad}\,(E_{kl} - E_{lk})Y_j) = \mathrm{ad}\,(E_{kl} - E_{lk})\tilde{J}(Y_j) = -\delta_{jl}X_k + \delta_{jk}X_l,\\
&\tilde{J}(\sqrt{-1}\mathrm{ad}\,(E_{kl} + E_{lk})Y_j) = \sqrt{-1}\mathrm{ad}\,(E_{kl} + E_{lk})\tilde{J}(Y_j)\\
&\qquad\qquad = \delta_{jl}Y_k + \delta_{jk}Y_l,\\
&\tilde{J}(\sqrt{-1}\mathrm{ad}\,(E_{kk} - E_{k+1k+1})Y_j) = \sqrt{-1}\mathrm{ad}\,(E_{kk} - E_{k+1k+1})\tilde{J}(Y_j)\\
&\qquad\qquad = \delta_{kj}Y_k - \delta_{jk+1}Y_{k+1},\\
&\tilde{J}(\mathrm{ad}\,(E_{kl} - E_{lk})V) = \mathrm{ad}\,(E_{kl} - E_{lk})\tilde{J}(V) = 0,\\
&\tilde{J}(\sqrt{-1}\mathrm{ad}\,(E_{kl} + E_{lk})V) = \sqrt{-1}\mathrm{ad}\,(E_{kl} + E_{lk})\tilde{J}(V) = 0,\\
&\tilde{J}(\sqrt{-1}\mathrm{ad}\,(E_{kk} - E_{k+1k+1})V) = \sqrt{-1}\mathrm{ad}\,(E_{kk} - E_{k+1k+1})\tilde{J}V = 0,
\end{aligned}
$$

where $V \in \{X_1, Y_1\}$. Hence, the tensor field J with $J_o = \tilde{J}$ on $(R^{2n}\backslash(0),\, g)$ is defined by

$$
J_{x\cdot o}(X) \;=\; (\tau_x)_\star \tilde{J}(({\tau_x)_\star}^{-1})X), \quad x \in G,\; X \in T_{x\cdot o}(R^{2n}\backslash(0)),
$$

where $(\tau_x)_\star$ is the differential of the translation $\tau_x \;:\; (R^{2n}\backslash(0)) \;\ni\; y \cdot o \longmapsto xy \cdot o \in (R^{2n}\backslash(0))$. From (2.3) and (3.7), we get

$$
N(X_b, X_a) \;=\; N(X_b, Y_a) \;=\; N(Y_b, Y_a) \;=\; 0. \tag{3.11}
$$

Therefore $(R^{2n}\backslash(0), g, J)$ is an n-dimensional complex manifold.

On the other hand, from (2.2) and (3.7) we obtain

$$U(X_b, X_a) = U(X_b, Y_a) = U(Y_b, Y_a) = 0. \tag{3.12}$$

Accordingly, $(R^{2n}\backslash(0), g)$ is a naturally reductive Riemannian homogeneous space. Since $\alpha(Y_k, Y_1) = -\sqrt{n/2(n-1)}X_k$, $(R^{2n}\backslash(0), g)$ is not symmetric. Hence, the proof of Theorem 5 is completed.

Moreover, we get from (2.1), (3.7) and (3.11)

$$\alpha(Y_1, X_1) = \alpha(X_k, X_1) = \alpha(Y_k, X_1) = 0. \tag{3.13}$$

Hence, the hypersurface (S^{2n-1}, g) of $(R^{2n}\backslash(0), g)$ is totally geodesic. Thus we get Theorem 4.

Remark 4. From (3.7), we get $d\omega(Y_1, X_k, Y_k) = 3\sqrt{2n/(n-1)}$. Hence, $(R^{2n}\backslash(0), g, J)$, $(n \geq 3)$, is not almost Kaehler.

Remark 5. $S^{n-1} = SO(n)/SO(n-1)$, $n \geq 2$; $G := R^{\times}SO(n)$;
$o := \{SO(n-1)\}$; $R^n\backslash(0) := R^{\times}SO(n)/SO(n-1)$;
$X_j := E_{1j} - E_{j1}$; $X_1 := I_n$;
$\mathfrak{m} := \{\sum k^b X_b \mid \text{each } k^b \in R\} = T_o(R^n\backslash(0))$.

We define an inner product $<\ ,\ >$ on $T_o(R^n\backslash(0))$ by

$$< X, Y > = \begin{cases} (-1/2)\ \text{trace}\,(XY), & X \in \mathfrak{m} \text{ and } Y \in \mathfrak{m}\backslash RI_n, \\ (1/n)\ \ \text{trace}\,(XY), & X, Y \in RI_n. \end{cases}$$

Let g be an G-invariant Riemannian metric on $R^n\backslash(0)$ determined by $<\ ,\ >$ on $T_o(R^n\backslash(0))$. Then, $(R^n\backslash(0), g)$ is a naturally reductive Riemannian homogeneous space of which the hypersurface S^{n-1} is totally geodesic. In fact, $\alpha(X_b, X_a) = 0$ and $U(X_b, X_a) = 0$.

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