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SOME FACTORIZATION PROPERTIES IN COMMUTATIVE RINGS

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Abstract: Some factorization properties are investigated in the context of commutative rings with zero divisors. Especially we introduce the notion of a finite decomposition ring (FDR) which is equivalent to an FFD for the domain case and by this definition each UFR is an FDR. Directed union of commutative rings having these factorization properties are also considered.

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1. Introduction

Let R be a commutative ring with identity. Any elements $a, b \in R$ are *associate*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, or equivalently $(a) = (b)$. A nonunit $a \in R$ is an *irreducible* if $a = bc \Rightarrow a \sim b$ or $a \sim c$. We allow 0 to be irreducible and 0 is irreducible $\Leftrightarrow$ R is an integral domain. R is called *atomic* if each nonzero, nonunit is a finite product of irreducible elements. R is said to be *presimplifiable* if $x = xy \Rightarrow x = 0$ or $y \in U(R)$. An integral domain or a quasilocal ring is a presimplifiable ring. A principal ideal ring (PIR) is called a *special principal ideal ring* (SPIR) if it has only one prime ideal $P \neq R$ and P

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is nilpotent. A new type of irreducible factorization in a commutative ring is a U-decomposition which is introduced by Fletcher [9]. A U-decomposition eliminates the bad behavior of a factorization because of nontrivial idempotents. A U-*decomposition* of $r \in R$ is a factorization $r = (p_1^{'}...p_k^{'})(p_1 \dots p_n)$ such that (i) the $p_i^{'}$'s and p_j's are irreducible; (ii) $p_i^{'}(p_1 \dots p_n) = (p_1 \dots p_n)$ for $i = 1, \dots, k$; and (iii) $p_i(p_1 \dots \hat{p}_i \dots p_n) \neq (p_1 \dots \hat{p}_i \dots p_n)$ for $i = 1, \dots, n$, where $\hat{p}_i$ means the omission of this element from the product. Let $r = (p_1^{'}...p_k^{'})(p_1 \dots p_n)$ be a U-decomposition of $r \in R$. Then the product $p_1 \dots p_n$ is called the *relevant part* and the other is the *irrelevant part*. Any irreducible decomposition can be rearranged to a U-decomposition. Two U-decompositions $r = (p_1^{'} \dots p_{k'}^{'})(p_1 \dots p_k) = (q_1^{'} \dots q_{n'}^{'})(q_1 \dots q_n)$ are *associates* if (i) $k = n$ and (ii) after a suitable change in the order of the factors in the relevant parts, we have p_i and q_i are associates for $i = 1, \dots, k$. R is called *unique factorization ring* (UFR) if each nonunit has a U-decomposition (or equivalently R is atomic) and any two U-decompositions of a nonunit element of R are associates. Fletcher shows that R is a UFR if and only if R is a finite direct product of UFD's and SPIR's.

Let D be an integral domain. In [2], some factorization properties which are weaker than unique factorization were introduced for an integral domain. D is a *bounded factorization domain* (BFD) if D is atomic and for each nonzero, nonunit of D there is a bound on the length of factorizations into products of irreducible elements. D is called a *half-factorial domain* (HFD) if D is atomic and each factorization of a nonzero, nonunit of D into a product of irreducible elements has the same length. HFD was introduced by Zaks in [13]. A domain D is a *finite factorization domain* (FFD) if each nonzero nonunit of D has only a finite number of nonassociate divisors and hence, only a finite number of factorizations up to order and associates. D is a FFD $\Leftrightarrow$ D is atomic and each nonzero element of D has only a finite number of nonassociate irreducible divisors.

These factorization properties are investigated for a commutative ring with zero divisors in a series of papers. For example see [4], [5], [1]. In this paper, we continue to investigate this properties in the context of commutative rings with zero divisors and especially we introduce the notion of a finite decomposition ring (FDR) which corresponds to an FFD in the domain case such that by this definition each UFR is an FDR.

2. Factorization Properties

Definition 1. Let R be a commutative ring with identity. R is a *finite decomposition ring* (FDR) if R is atomic and if every nonunit has only a finite number of nonassociate U-decompositions.

If R is an integral domain then R is an FDR if and only if R is an FFD. So the notion of an FDR is a generalization of the notion of an FFD to commutative rings with zero divisors. It is clear that by this definition we have the implication

$$\text{UFR} \Rightarrow \text{FDR}$$

as in the domain case.

In [4], the authors gave a family of definitions which are equivalent to FFD for the case R is an integral domain. A commutative ring R is a *finite factorization ring* (FFR) if every nonzero nonunit of R has only a finite number of factorizations up to order and associates; R is called a *weak finite factorization ring* (WFFR) if every nonzero nonunit of R has only a finite number of nonassociate divisors; and R is called an *atomic* idf-*ring* if R is atomic and each nonzero element of R has at most a finite number of nonassociate irreducible divisors. R is an FFR $\Rightarrow$ R is a WFFR $\Rightarrow$ R is an atomic idf-ring. But non of this implication can be reversed [4].

Let $R = Z \times Z$. Then R is clearly a UFR and hence an FDR. Consider the element $(0,6)$. Then $(0,6) = (p,1)(1,2)(0,3)$ is an infinitely many distinct factorization for every prime integer p. Thus R is not even an atomic idf-ring. So R is a UFR need not imply R is an FFR or a WFFR or an atomic idf-ring.

Later Axtell [6] gave a generalization of the notion of an FFD to commutative rings by using U-factorization. Let $r \in R$ be a nonunit and let $r = a_1 \ldots a_n b_1 \ldots b_m$ is a factorization of r into a product of nonunit elements. Then $r = (a_1 \ldots a_n)(b_1 \ldots b_m)$ is a *U-factorization* of r if (i) $a_i(b_1 \ldots b_m) = (b_1 \ldots b_m)$ for $1 \leq i \leq n$, and (ii) $b_j(b_1 \ldots \tilde{b}_j \ldots b_m) \neq (b_1 \ldots \tilde{b}_j \ldots b_m)$ for $1 \leq j \leq m$. U-factorization is introduced in [1]. A ring R is a *U-finite factorization ring* (U-FFR) if every nonzero nonunit of R has only a finite number of U-factorizations up to order and associates on the essential divisors. But this definition does not also yield the implication R is a UFR $\Rightarrow$ R is a U-FFR. $(0,6) = (\,)((k,2)(0,3))$ is infinitely many distinct U-factorization of $(0,6)$ in $Z \times Z$ for every $k \in Z$.

Example 2. *An atomic idf-ring R which is not an FDR. Let (R, M) be a quasilocal ring with M is an infinite and $M^2 = 0$ (e.g. $R = Z_4 + X(2)[X]$). Then each nonzero nonunit element of R is an irreducible. So R is an atomic idf-ring. But $0 = (\,)(a_1 a_2)$ for every nonzero $a_1, a_2 \in M$ and so R is not an FDR.*

Theorem 3. (Dickson's Theorem) *Let $S = N \times \ldots \times N$, the direct cartesian product of n copies of N. Let $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ in S if and only if $a_i \leq b_i$ for $1 \leq i \leq n$; this is a partial ordering on S. If $\emptyset \neq X \subseteq S$, then X has only finitely many minimal elements [8].*

We say that $a \in R$ is U-$finite$ if a has only finitely many nonassociate U-decompositions.

Theorem 4. *Let R be a commutative ring with identity. If R is an atomic idf-ring and 0_R is U-finite then R is an FDR.*

Proof. Let $x \in R$ be a nonzero nonunit and let $\{q_1, \dots, q_n\}$ be the set of entire list of nonassociate irreducible divisors of x. Any U-decomposition of x is of the form $x = (q_1^{b_1} \dots q_n^{b_n})(q_1^{a_1} \dots q_n^{a_n})$, where a_i's and b_i's are nonnegative integers. Consider the set $X = \{(a_1, \dots, a_n) \mid x = (q_1^{c_1} \dots q_n^{c_n})(q_1^{a_1} \dots q_n^{a_n})$ is a U-decomposition of $x\}$. Since R is atomic X is not empty. Let the partial ordering be given by the above theorem. Let $(a_i), (b_i) \in X$ such that $(a_i) \leq (b_i)$. If $(a_i) \neq (b_i)$ then $a_i < b_i$ for some i, say $a_1 < b_1$. So $(x) = (q_1^{b_1} \dots q_n^{b_n}) \subsetneqq (q_1^{a_1} q_2^{b_2} \dots q_n^{b_n}) = (q_2^{b_2 - a_2} \dots q_n^{b_n - a_n})(q_1^{a_1} \dots q_n^{a_n}) \subseteq (x)$, a contradiction. Thus each element of X is minimal in X. Hence by Dickson's Theorem X is finite. Thus R is an FDR. □

Theorem 5. *If any commutative ring R with identity is presimplifiable and an FDR then R is an FFR.*

Proof. Since R is presimplifiable in any U-decomposition of nonzero element the irrelevant part is empty. Thus R is an atomic idf-ring. Then R is an FFR since R is an FFR if and only if R is presimplifiable and an atomic idf-ring [4, Proposition 6.6]. □

Let R be an atomic ring and $x \in R$ be a non unit. Let us denote the number of distinct U-decompositions of x by $Nd(x)$. If x is a unit then we define $Nd(x) = 1$. Thus for any non unit x, $Nd(x) = 1 \Leftrightarrow x$ is irreducible.

Theorem 6. *Let R_1, R_2 be commutative rings which are atomic and $R = R_1 \times R_2$. Then for any element $a = (a_1, a_2)$*

$$Nd(a) = Nd(a_1).Nd(a_2).$$

Proof. If a is a unit in R then each a_i is a unit in R_i. Hence $Nd(a) = Nd(a_i) = 1$. Suppose a is not a unit. It is clear that some a_i has infinitely many distinct U-decomposition in R_i if and only if a has infinitely many distinct U-decomposition in R. So we may assume $Nd(a_1) = m$ and $Nd(a_2) = n$. For convenient we take the irrelevant part of any U-decomposition is empty. Suppose the entire nonassociate U-decompositions of a_1 in R_1 and a_2 in R_2 as follows:

$$a_1 = (\,)(p_{11} \dots p_{1k_1}) = (\,)(p_{21} \dots p_{2k_2}) = \dots = (\,)(p_{m1} \dots p_{mk_m}),$$

$$a_2 = (\,)(q_{11} \dots q_{1l_1}) = (\,)(q_{21} \dots p_{2l_2}) = \dots = (\,)(q_{n1} \dots p_{nl_n}).$$

Let us consider $a_1 = (\,)(p_{11} \ldots p_{1k_1})$ the first U-decomposition of a_1 in R_1. By denoting $\tilde{p}_{ij} = (p_{ij}, 1)$ and $\tilde{q}_{ij} = (1, q_{ij})$, a has n number distinct U-decompositions in R as

$$a = (\,)(\tilde{p}_{11} \ldots \tilde{p}_{1k_1}\tilde{q}_{11} \ldots \tilde{q}_{1l_1}) = (\,)(\tilde{p}_{11} \ldots \tilde{p}_{1k_1}\tilde{q}_{21} \ldots \tilde{q}_{2l_2}) = \ldots$$
$$= (\,)(\tilde{p}_{11} \ldots \tilde{p}_{1k_1}\tilde{q}_{n1} \ldots \tilde{q}_{nl_n}) .$$

By repeating the same argument for m distinct U-decompositions of a_1, we have $m.n$ number distinct U-decomposition of a in R. Now we show that any U-decompositions of a in R is one of the U-decompositions of a. Let $a = (a_1, a_2) = (\,)((t_1, s_1) \ldots (t_\alpha, s_\alpha))$ be any U-decomposition of a in R. Since any irreducible element of $R_1 \times R_2$ is unit in one coordinate and irreducible in the other. So we may take $t_k = t_{k+1} = \ldots = t_\alpha = 1$ and $s_1 = s_2 = \ldots = s_{k-1} = 1$. Thus $a_1 = (\,)(t_1 \ldots t_{k-1})$ is one of the U-decompositions of a_1 in R_1 and $a_2 = (\,)(s_k \ldots s_\alpha)$ is one of the U-decompositions of a_2 in R_2. Therefore any U-decomposition of a is an associate of one of the above U-decompositions of a. Hence $Nd(a) = m.n = Nd(a_1).Nd(a_2)$. □

Corollary 7. *Let $R_1, \ldots, R_n$ be commutative rings which are atomic and let $R = R_1 \times \ldots \times R_n$. Then for any element $a = (a_1, \ldots, a_n)$ we have $Nd(a) = Nd(a_1) \cdots Nd(a_n)$.*

It is known that $R_1 \times \ldots \times R_n$ is atomic if and only if each R_i is atomic [4]. Thus we have the following corollary.

Corollary 8. *$R_1 \times \ldots \times R_n$ is a UFR if and only if each R_i is a UFR, [9].*

Theorem 9. *Let R be commutative ring. Then R is an FDR if and only if R is a finite direct product of FDR's.*

Proof. If R is an FDR then 0 is a finite product of irreducible elements. Therefore $R = R_1 \times \ldots \times R_n$ a finite product of indecomposable rings [4]. So the proof follows from Corollary 7. □

Corollary 10. *R is a FDR if and only if R is a finite direct product of FFD's, FFR's with 0 is U-finite and indecomposable FDR's with proper zero divisors.*

Proof. ($\Rightarrow$) Let $R = R_1 \times \ldots \times R_n$ with each R_i is an indecomposable FDR. If R_i is a domain then R_i is an FFD. Suppose R_i is not a domain. If R_i is presimplifiable then by Theorem 5, R_i is an FFR.

($\Leftarrow$) Any FFR is an atomic idf-ring. Hence by Theorem 4, an FFR in which 0 is U-finite is an FDR. □

Now we briefly discuss the other factorization properties. Let us call a commutative ring R with identity to be a *half decomposition ring* (HDR) if R is atomic and for each nonunit $x \in R$ if $x = (p_1^{'} ... p_k^{'})(p_1 ... p_m) = (q_1^{'} ... q_l^{'})(q_1 ... q_n)$ are two U-decompositions of x then $m = n$.

Theorem 11. *Let R be a commutative ring with identity and $R = R_1 \times R_2$. Then R is an HDR if and only if R_1 and R_2 are HDR's.*

Proof. ($\Rightarrow$) It is enough to show that R_1 is an HDR. Let $a_1 \in R_1$ be a nonunit and $a_1 = (q_1^{'}q_2^{'} ... q_k^{'})(q_1q_2 ... q_n) = (p_1^{'}p_2^{'} ... p_l^{'})(p_1p_2 ... p_m)$ are two U-decompositions of a_1 in R_1. Take $\tilde{r} = (r, 1) \in R_1 \times R_2$. Since irrelevant parts play no role in verifying the half decomposition property, we may take the irrelevant parts are empty. Then $(\,)(\tilde{q}_1\tilde{q}_2 ... \tilde{q}_n) = (\,)(\tilde{p}_1\tilde{p}_2 ... p_m)$ are two U-decompositions of $\tilde{a}_1 \in R$ and $m = n$.

($\Leftarrow$) Let $a = (a_1, a_2) \in R$ be a non unit. Since an irreducible element of $R_1 \times R_2$ is a unit in one coordinate and an irreducible in the other [4], we may take any two U-decompositions of a as

$$\begin{aligned} a = (\,)((q_1,1)(q_2,1) ... (q_k,1)(1,t_1)(1,t_2) ... (1,t_n)) \\ = (\,)((p_1,1)(p_2,1) ... (p_l,1)(1,s_1)(1,s_2) ... (1,s_m)). \end{aligned}$$

Then we have U-decompositions in R_1, $a_1 = (\,)(q_1q_2 ... q_k) = (\,)(p_1p_2 ... p_l)$ and in R_2, $a_2 = (\,)(t_1t_2 ... t_n) = (\,)(s_1s_2 ... s_m)$. So $k = l$, $n = m$ and hence $k + n = l + m$. □

Corollary 12. *Let R be a commutative ring with identity. Then R is an HDR if and only R is a finite direct product of HDR's. In particular a finite direct product of HFD's is an HDR.*

Definition 13. Let R be a commutative ring with identity. R is a bounded decomposition ring (BDR) if R is atomic and for each nonunit $x \in R$, there is a positive integer $U - N(x)$ such that whenever $x = (p_1^{'} ... p_k^{'})(p_1 ... p_n)$ as a U-decomposition of x then $n < U\text{-}N(x)$.

It is clear that if R is an integral domain then R is a BDR if and only if R is a BFD. So the notion of BDR is a extension of the notion of BFD to commutative rings with zero divisors. In [4], authors give another definition. A commutative ring R is called a bounded factorization ring (BFR) if for each nonzero nonunit $a \in R$, there exist a natural number $N(a)$ so that for any factorization $a = a_1 ... a_n$ of a, where each a_i is a nonunit we have $n < N(a)$. Another definition is due to Ağargün, et al. R is a U-bounded factorization ring (U-BFR) if every nonzero element of R is U-bounded. If R is a BFR or a BDR then R is a U-BFR. But there is no other implication possible among these rings, in general. In [1] it is given an example of a BFR R with 0 is not U-finite [1, Example 4.2]. Hence R is a U-BFR or a BFR need not imply R is

BDR. For a detailed study of the notions a BFR and a U-BFR, see [1]. But for a presimplifiable ring, we have the following equality.

Proposition 14. *Let R be a commutative ring. If R is presimplifiable then the following conditions are equivalent.*

1. *R is a BDR.*
2. *R is a BFR and is 0 U-bounded.*
3. *R is a U-BFR and is 0 U-bounded.*

Proof. Since R is presimplifiable, in any U-decomposition of any nonzero nonunit element $r \in R$, the irrelevant part is empty. So any irreducible decomposition of r is also a U-decomposition of r. Hence $1 \Leftrightarrow 2, 3$ Equivalency of 2 and 3 also comes from [1, Theorem 4.2]. □

Following theorem is in [1].

Theorem 15. *Let $R_1, \ldots, R_n$ be commutative rings, $n > 1$, and let $R = R_1 \times \ldots \times R_n$. Then R is a U-BFR $\Leftrightarrow$ each R_i is a U-BFR and 0_{R_i} is U-bounded. Hence 0_R is U-bounded.*

Theorem 16. *Let R be commutative ring. Then R is a BDR if and only if R is a finite direct product of BDR's.*

Proof. Let R be a BDR. Since 0 is a finite product of irreducible elements, R is a finite direct product of indecomposable rings, say $R = R_1 \times \ldots \times R_n$. Then R is atomic if and only if each R_i is atomic. Hence the second condition of BDR is follows from Theorem 15. □

Corollary 17. *R is a BDR if and only if R is a finite direct product of BFD's, BFR's with 0 is U-bounded and indecomposable BDR's with proper zero divisors.*

Proof. It is enought to show only the implication $\Rightarrow$. Let $R = R_1 \times \ldots \times R_n$ with each R_i is an indecomposable BDR. If R_i is an integral domain then R_i is a BFD. Suppose R_i is not a domain. If R_i is presimplifiable then R_i is a BFR in which 0 is U-bounded. □

3. Directed Union

Recall that an extension $A \subseteq B$ of commutative rings is a (*weakly*) *inert extension* if whenever ($0 \neq xy \in A$) $xy \in A$ for nonzero $x, y \in B$, then $xu, u^{-1}y \in A$ for some $u \in U(B)$ [1]. Weak form of this definition is due to Ağargün et al, see [1]. Clearly an inert extension is a weakly inert extension, but the converse is not true in general. Let $A \subseteq B$ be a weakly inert extension of commutative rings. If $0 \neq a \in A$ is irreducible, then as an element of B, either a is irreducible or a is a unit [1, Proposition 2.1]. But if $A \subseteq B$ is an inert extension then we may take $a = 0$ in this proposition. For if 0 is an irreducible in A then A is a domain. If $0 = xy \in B$ then $0 = xu.u^{-1}y$ in A. Thus xu or $u^{-1}y$ is 0, and hence $x = 0$ or $y = 0$.

For the case of integral domains, factorization properties of directed unions of commutative rings $\{R_\alpha\}$, where each $R_\alpha \subset R_\beta$ is an inert extension were investigated in [3]. In what follows we extend these results to commutative rings with zero divisors.

Lemma 18. *Let $\{R_\alpha\}$ be a directed family of atomic commutative rings such that each $R_\alpha \subset R_\beta$ is an inert extension and let $R = \cup R_\gamma$. If $x \in R$ is irreducible than x is irreducible in some R_γ.*

Proof. Let $x \in R_\alpha$. Since R_α is atomic, $x = x_1 \dots x_n$, where each $x_i \in R_\alpha$ is irreducible. Since x is irreducible in R, x is an associate of one of the x_i's in R, say x_n. Then $x_n = xr$, for some $r \in R$. Let $r \in R_\gamma$. If $R_\gamma \subset R_\alpha$ then $r \in R_\alpha$ and $x \sim x_n$ in R_α. Since any associate element of an irreducible element is also an irreducible [4], x is an irreducible element of R_α. Suppose R_γ is not a subset of R_α. Then there exist $R_\beta \in \{R_\alpha\}$ such that $R_\alpha \subset R_\beta$ and $R_\gamma \subset R_\beta$. Hence x_n is either irreducible or a unit in R_β. Since x is not a unit in R_β and $x_n = xr$, x_n cannot be a unit in R_β. Hence x_n is irreducible in R_β. So x is irreducible in R_β, since x is an associate of x_n in R_β. □

Lemma 19. *Let $\{R_\alpha\}$ be a directed family of atomic commutative rings such that each $R_\alpha \subset R_\beta$ is an inert extension and let $R = \cup R_\gamma$. If for any non unit $x \in R$, $x = (q_1 \dots q_m)(p_1 \dots p_n)$ is a U-decomposition of x in R, then this is also a U-decomposition of x in some R_γ.*

Proof. Let $x = (q_1 \dots q_m)(p_1 \dots p_n)$ be a U-decomposition of x in R. By Lemma 18, the elements q_i and p_j are also irreducibles in some R_α for $i = 1, \dots, m$, $j = 1, \dots, n$. Thus q_i's and p_j's are remain irreducibles in all $R_\beta \supset R_\alpha$ since q_i and p_j are not units in R. Since $x = (q_1 \dots q_m)(p_1 \dots p_n)$ is U-decomposition of x in R, $q_i(p_1 \dots p_n) = (p_1 \dots p_n)$ for $i = 1, \dots, m$. Therefore there exist some $r_i \in R$ such that $p_1 \dots p_n = p_1 \dots p_n q_i r_i$ for $i = 1, \dots, m$. If $\forall r_i \notin R_\alpha$ then for some $R_\beta \supset R_\alpha$, $r_i \in R_\beta$ for $i = 1, \dots, m$. So in R_β we have the equality of ideals $q_i(p_1 \dots p_n) = (p_1 \dots p_n)$. For the last condition of U-decoposition, suppose $(p_1 \dots p_{j-1} p_{j+1} \dots p_n) = (p_1 \dots p_n)$ in R_β for $j = 1, \dots, n$. But in this case, this equality also holds in R which contradicts the fact $x = (q_1 \dots q_m)(p_1 \dots p_n)$ is a U-decomposition of x in R. Hence $x = (q_1 \dots q_m)(p_1 \dots p_n)$ is also a U-decomposition of x in some R_β. □

Theorem 20. *Let $\{R_\alpha\}$ be a directed family of atomic commutative rings* *such that each $R_\alpha \subset R_\beta$ is an inert extension and let $R = \cup R_\gamma$.*

i) R is atomic if each R_γ is atomic.

ii) R satisfies ACCP if each R_γ satisfies ACCP.

iii) R is a BDR if each R_γ is a BDR.

iv) R is a HDR if each R_γ is a HDR.

v) R is a FDR if each R_γ is an FDR.

Proof. i) See [1, Proposition 2.5].

ii) Let $aR \subset bR$ be a strictly increasing chain in R. Let $a \in R_\alpha$. Then $a = br$ for some $r \in R$. It is easily seen that $R_\alpha \subset R$ is an inert extension. So bu, $ru^{-1} \in R_\alpha$ for some $u \in U(R)$. Let $b' = bu$. Then $aR \subset b'R$ is a strictly increasing a chain in R and $a, b' \in R_\alpha$. So $aR_\alpha \subset b'R_\alpha$. Hence for each strictly increasing chain of principal ideals of length n in R, we have a chain of principal ideals of length n in some R_α. Thus if each R_α satisfies ACCP then so R does.

iii)–iv) From (i), R is atomic. Let x be any element in R. By Lemma 19, if $x = (q_1 ... q_m)(p_1 ... p_n)$ is a U-decomposition of x in R then this U-decomposition is also valid for x in some R_α which is a BDR (an HDR). Hence R is a BDR (an HDR).

v) Observe that if any two element in some R_α is associate in R_α than they are assosiate in R. Hence any two distinct U-decompositions of $x \in R$ has remain distinct for some R_α. So if each R_α is an FDR then R_α is an FDR. □

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