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SEMIGROUP METHOD FOR M/G/1 RETRIAL QUEUE WITH GENERAL RETRIAL TIMES

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Abstract: In this paper, we set up M/G/1 retrial queueing model with general retrial times as an abstract Cauchy problem and prove the existence of a unique positive time-dependent solution by using C_0- semigroup theory of linear operators.

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1. Introduction

Queueing systems in which arriving customers who find all servers and waiting positions occupied may retry for service after a period of time are called retrial queues or queues with repeated orders. Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication systems, computer networks and computer systems. Because of strong practical background and the wide application, there has been a rapid growth in the literature on retrial queueing systems (see Choi [1], Gomez-Corel [4], Yang [9]). Because of the complexity of retrial queueing models, time-dependent results are scarce in literatures. In contrast, there are many results about steady-state solutions. In [4] the author established an M/G/1 retrial queueing model with

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general retrial times by using supplementary variable technique and obtained the existence of its steady-state solution by using probability generating function. In [10] the author explored a stochastic decomposition property of an M/G/1 retrial queue with general retrial times and proposed an approximation method for the calculation of the steady-state performance measures of the system. In [5] the author considered a generalization of the classical Erlang loss model with both retrials of blocked calls and a time-dependent arrival rate. Under exponential distribution assumptions, they obtained the time-dependent mean number of calls in progress and the times of peak blocking.

It is difficult to get results about time-dependent solutions in the usual methods such as probability generating function, Brownian motion and Laplace transform, etc. In this paper, with the aid of C_0- semigroup theory of linear operators we prove the existence of a unique positive time-dependent solution for M/G/1 retrial queueing model with general retrial times. During reading this paper, readers will find that our method can be used to many retrial queueing models.

M/G/1 retrial queueing model with general retrial times can be expressed as (see Gomez-Coral [4])

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \int_0^\infty Q_0(x,t)s(x)dx, \tag{1}$$

$$\frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial x} = -(\lambda + r(x))p_n(x,t), \quad n \geq 1, \tag{2}$$

$$\frac{\partial Q_0}{\partial t} + \frac{\partial Q_0}{\partial x} = -(\lambda + s(x))Q_0(x,t), \tag{3}$$

$$\frac{\partial Q_n}{\partial t} + \frac{\partial Q_n}{\partial x} = -(\lambda + s(x))Q_n(x,t) + \lambda Q_{n-1}(x,t), \quad n \geq 1, \tag{4}$$

$$p_n(0,t) = \int_0^\infty Q_n(x,t)s(x)dx, \quad n \geq 1, \tag{5}$$

$$Q_0(0,t) = \lambda p_0(t) + \int_0^\infty p_1(x,t)r(x)dx, \tag{6}$$

$$Q_n(0,t) = \lambda \int_0^\infty p_n(x,t)dx + \int_0^\infty p_{n+1}(x,t)r(x)dx, \quad n \geq 1, \tag{7}$$

$$p_0(0) = 1, \quad p_n(x,0) = 0, \quad n \geq 1, \quad Q_j(x,0) = 0, \quad j \geq 0. \tag{8}$$

Here $p_0(t)$ represents the probability that at time t there is no customer in orbit and the server is idle. $p_n(x,t)dx$ $(n \geq 1)$ represents the probability that at time t the server is idle and there are n customers in orbit with elapsed retrial time lying in $[x, x+dx)$. $Q_n(x,t)dx$ represents the probability that at time t the server is busy and there are n customers in orbit with the elapsed service time lying $[x, x+dx)$. λ represents the arrival rate of customers. $r(x)$ and $s(x)$ are the conditional completion rates for repeated attempt and for service at x, respectively. $s(x)$ and $r(x)$ satisfy

$$r(x) \geq 0, \quad \int_0^\infty r(x)dx = \infty, \quad s(x) \geq 0, \quad \int_0^\infty s(x)dx = \infty.$$

Throughout this paper we assume that $s(x)$ and $r(x)$ satisfy

$$\alpha = \sup_{x\in[0,\infty)} s(x) < \infty, \quad \mu = \sup_{x\in[0,\infty)} r(x) < \infty.$$

Remark 1. (8) means that at time $t = 0$ there is no customer in orbit and the server is idle. In other words, $\lambda = 0$, $s(x) = r(x) = 0$. Thus at the point $(0,0)$ (i.e. $x = 0$, $t = 0$), (5), (6) and (7) become $p_n(0,0) = Q_n(0,0) = 0$, that is, the above equations (1)–(8) satisfy compatibility condition. We see later (Theorem 2) this property is important for us.

Choose the state space as follows

$$X = \{g = (p, Q) \mid \|g\| = \|p\| + \|Q\| < \infty\},$$

where

$$p \in R \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots,$$
$$Q \in L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots,$$
$$\|p\| = |p_0| + \sum_{n=1}^\infty \|p_n\|_{L^1[0,\infty)}, \quad \|Q\| = \sum_{n=0}^\infty \|Q_n\|_{L^1[0,\infty)}.$$

It is obvious that X is a Banach space. For simplicity, we introduce some notations as follows.

$$\Gamma_0 = \begin{pmatrix} e^{-x} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & s(x) & 0 & 0 & \cdots \\ 0 & 0 & s(x) & 0 & \cdots \\ 0 & 0 & 0 & s(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} \lambda e^{-x} & r(x) & 0 & 0 & 0 & \cdots \\ 0 & \lambda & r(x) & 0 & 0 & \cdots \\ 0 & 0 & \lambda & r(x) & 0 & \cdots \\ 0 & 0 & 0 & \lambda & r(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In the following we define operators and their domains.

$$A(p,Q) = \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\frac{d}{dx} & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \Big\{ (p,Q) \in X \,\Big|\, \frac{dp_i(x)}{dx} \in L^1[0,\infty),\ (i \geq 1), \frac{dQ_n(x)}{dx} \in L^1[0,\infty),$$
$(n \geq 0), p_i(x)$ and $Q_n(x)$ are absolutely continuous functions and satisfy
$$p(0) = \int_0^\infty \Gamma_0 p(x)dx + \int_0^\infty \Gamma_1 Q(x)dx, \quad Q(0) = \int_0^\infty \Gamma_2 p(x)dx \Big\}.$$

$$U(p,Q) =$$
$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -(\lambda + r(x)) & 0 & 0 & \cdots \\ 0 & 0 & -(\lambda + r(x)) & 0 & \cdots \\ 0 & 0 & 0 & -(\lambda + r(x)) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix}, \right.$$

$$\begin{pmatrix} -(\lambda + s(x)) & 0 & 0 & 0 & 0 & \cdots \\ \lambda & -(\lambda + s(x)) & 0 & 0 & 0 & \cdots \\ 0 & \lambda & -(\lambda + s(x)) & 0 & 0 & \cdots \\ 0 & 0 & \lambda & -(\lambda + s(x)) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$
$$\left. \times \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix} \right),$$

$$E(p,Q)=\begin{pmatrix}\int_0^\infty Q_0(x)s(x)dx\\ 0\\ 0\\ \vdots\end{pmatrix},\quad D(U)=X,\quad D(E)=X.$$

Then the above system of equations (1)–(8) can be written as an abstract Cauchy problem in the Banach space X

$$\frac{d(p,Q)(t)}{dt}=(A+U+E)(p,Q)(t),\quad t\in[0,\infty), \tag{9}$$

$$(p,Q)(0)=(p(0),Q(0)), \tag{10}$$

$$p(0)=(1,0,0,\cdots),\quad Q(0)=(0,0,0,\cdots).$$

2. Main Results

Theorem 1. *$A+U+E$ generates a positive contraction C_0- semigroup $T(t)$.*

Proof. For clarity, we will split the proof of this theorem into four steps. Firstly, we prove that $(\gamma I-A)^{-1}$ exists and is bounded for some γ. Secondly, we prove that $D(A)$ is dense in X. Thus by using the Hille-Yosida Theorem we obtain that A generates a C_0- semigroup. Thirdly, we show that U and E are bounded linear operators. Thus by using the perturbation theory of C_0- semigroup we deduce that $A+U+E$ generates a C_0- semigroup $T(t)$. Finally, we verify that $A+U+E$ is dispersive. This allows us to use the Phillips Theorem and to get that $T(t)$ is positive contractive.

For any given $(y,z)\in X$ we consider the equation $(\gamma I-A)(p,Q)=(y,z)$. This is equivalent to

$$(\gamma+\lambda)p_0=y_0, \tag{11}$$

$$\gamma p_n(x)+\frac{dp_n(x)}{dx}=y_n(x),\quad n\geq 1, \tag{12}$$

$$\gamma Q_n(x)+\frac{dQ_n(x)}{dx}=z_n(x),\quad n\geq 0, \tag{13}$$

$$p_n(0)=\int_0^\infty Q_n(x)s(x)dx,\quad n\geq 1, \tag{14}$$

$$Q_0(0)=\int_0^\infty p_1(x)r(x)dx+\lambda p_0, \tag{15}$$

$$Q_n(0) = \lambda \int_0^\infty p_n(x)dx + \int_0^\infty p_{n+1}(x)r(x)dx, \quad n \geq 1. \tag{16}$$

Solving (11)–(13), we have

$$p_0 = \frac{1}{\gamma + \lambda} y_0, \tag{17}$$

$$p_n(x) = a_n e^{-\gamma x} + e^{-\gamma x} \int_0^x y_n(\tau) e^{\gamma \tau} d\tau, \quad n \geq 1, \tag{18}$$

$$Q_n(x) = b_n e^{-\gamma x} + e^{-\gamma x} \int_0^x z_n(\tau) e^{\gamma \tau} d\tau, \quad n \geq 0. \tag{19}$$

Combining (18) and (19) with (14), we deduce

$$\begin{aligned} a_n = p_n(0) = &\int_0^\infty Q_n(x)s(x)dx \\ &= \int_0^\infty [b_n e^{-\gamma x} + e^{-\gamma x} \int_0^x z_n(\tau) e^{\gamma \tau} d\tau] s(x)dx \\ = b_n &\int_0^\infty e^{-\gamma x} s(x)dx + \int_0^\infty e^{-\gamma x} s(x) \int_0^x z_n(\tau) e^{\gamma \tau} d\tau dx, \quad n \geq 1. \end{aligned} \tag{20}$$

Combining (15), (17), (18) with (19) we obtain that

$$\begin{aligned} b_0 = Q_0(0) = &\int_0^\infty p_1(x)r(x)dx + \lambda p_0 = a_1 \int_0^\infty e^{-\gamma x} r(x)dx \\ &+ \int_0^\infty e^{-\gamma x} r(x) \int_0^x y_1(\tau) e^{\gamma \tau} d\tau dx + \frac{\lambda}{\gamma + \lambda} y_0. \end{aligned} \tag{21}$$

Combining (18) and (19) with (16) and change order of the integral, it follows that

$$\begin{aligned} b_n = Q_n(0) = &\lambda \int_0^\infty p_n(x)dx + \int_0^\infty p_{n+1}(x)r(x)dx \\ &= \frac{\lambda}{\gamma} a_n + \lambda \int_0^\infty e^{-\gamma x} \int_0^x y_n(\tau) e^{\gamma \tau} d\tau dx \\ &+ a_{n+1} \int_0^\infty r(x) e^{-\gamma x} dx + \int_0^\infty r(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma \tau} d\tau dx \\ = &\frac{\lambda}{\gamma} a_n + a_{n+1} \int_0^\infty r(x) e^{-\gamma x} dx + \lambda \int_0^\infty y_n(\tau) e^{\gamma \tau} \int_\tau^\infty e^{-\gamma x} dx d\tau \\ &+ \int_0^\infty r(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma \tau} d\tau dx \end{aligned}$$

$$= \frac{\lambda}{\gamma} a_n + a_{n+1} \int_0^\infty r(x) e^{-\gamma x} dx + \frac{\lambda}{\gamma} \int_0^\infty y_n(\tau) d\tau + \int_0^\infty r(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma \tau} d\tau dx, \quad n \geq 1. \quad (22)$$

From (20) and (22) we know that

$$\begin{aligned} b_n = & \frac{\lambda}{\gamma} \{ b_n \int_0^\infty e^{-\gamma x} s(x) dx + \int_0^\infty e^{-\gamma x} s(x) \int_0^x z_n(\tau) e^{\gamma \tau} d\tau dx \} \\ & + \int_0^\infty r(x) e^{-\gamma x} dx \{ b_{n+1} \int_0^\infty e^{-\gamma x} s(x) dx \\ & + \int_0^\infty s(x) e^{-\gamma x} \int_0^x z_{n+1}(\tau) e^{\gamma \tau} d\tau dx \} \\ & + \frac{\lambda}{\gamma} \int_0^\infty y_n(\tau) d\tau + \int_0^\infty r(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma \tau} d\tau dx \\ = & b_n \frac{\lambda}{\gamma} \int_0^\infty e^{-\gamma x} s(x) dx + b_{n+1} \int_0^\infty r(x) e^{-\gamma x} dx \int_0^\infty e^{-\gamma x} s(x) dx \\ & + \frac{\lambda}{\gamma} \int_0^\infty e^{-\gamma x} s(x) \int_0^x z_n(\tau) e^{\gamma \tau} d\tau dx \\ & + (\int_0^\infty r(x) e^{-\gamma x} dx) \int_0^\infty e^{-\gamma x} s(x) \int_0^x z_{n+1}(\tau) e^{\gamma \tau} d\tau dx \\ & + \frac{\lambda}{\gamma} \int_0^\infty y_n(\tau) d\tau + \int_0^\infty r(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma \tau} d\tau dx, \quad n \geq 1. \quad (23) \end{aligned}$$

If we set

$$C = \begin{pmatrix} 1-\frac{\lambda}{\gamma}\omega_1 & -\omega_2\omega_1 & 0 & 0 & 0 & \cdots \\ 0 & 1-\frac{\lambda}{\gamma}\omega_1 & -\omega_2\omega_1 & 0 & 0 & \cdots \\ 0 & 0 & 1-\frac{\lambda}{\gamma}\omega_1 & -\omega_2\omega_1 & 0 & \cdots \\ 0 & 0 & 0 & 1-\frac{\lambda}{\gamma}\omega_1 & -\omega_2\omega_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $\omega_1 = \int_0^\infty e^{-\gamma x} s(x) dx$ and $\omega_2 = \int_0^\infty e^{-\gamma x} r(x) dx$,

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix},$$

then (23) can be rewritten as

$$C\vec{b}=\begin{pmatrix}\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_1(\tau)e^{\gamma\tau}d\tau dx+\omega_2\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_2(\tau)e^{\gamma\tau}d\tau dx\\ \frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_2(\tau)e^{\gamma\tau}d\tau dx+\omega_2\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_3(\tau)e^{\gamma\tau}d\tau dx\\ \frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_3(\tau)e^{\gamma\tau}d\tau dx+\omega_2\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_4(\tau)e^{\gamma\tau}d\tau dx\\ \vdots\\ +\frac{\lambda}{\gamma}\int_0^\infty y_1(x)dx+\int_0^\infty r(x)e^{-\gamma x}\int_0^x y_2(\tau)e^{\gamma\tau}d\tau dx\\ +\frac{\lambda}{\gamma}\int_0^\infty y_2(\tau)d\tau+\int_0^\infty r(x)e^{-\gamma x}\int_0^x y_3(\tau)e^{\gamma\tau}d\tau dx\\ +\frac{\lambda}{\gamma}\int_0^\infty y_3(x)dx+\int_0^\infty r(x)e^{-\gamma x}\int_0^x y_4(\tau)e^{\gamma\tau}d\tau dx\\ \vdots\end{pmatrix}. \tag{24}$$

It is easy to calculate that

$$C^{-1}=\begin{pmatrix}\frac{1}{1-\frac{\lambda}{\gamma}\omega_1} & \frac{\omega_1\omega_2}{(1-\frac{\lambda}{\gamma}\omega_1)^2} & \frac{(\omega_1\omega_2)^2}{(1-\frac{\lambda}{\gamma}\omega_1)^3} & \frac{(\omega_1\omega_2)^3}{(1-\frac{\lambda}{\gamma}\omega_1)^4} & \cdots\\ 0 & \frac{1}{1-\frac{\lambda}{\gamma}\omega_1} & \frac{\omega_1\omega_2}{(1-\frac{\lambda}{\gamma}\omega_1)^2} & \frac{(\omega_1\omega_2)^2}{(1-\frac{\lambda}{\gamma}\omega_1)^3} & \cdots\\ 0 & 0 & \frac{1}{1-\frac{\lambda}{\gamma}\omega_1} & \frac{\omega_1\omega_2}{(1-\frac{\lambda}{\gamma}\omega_1)^2} & \cdots\\ 0 & 0 & 0 & \frac{1}{1-\frac{\lambda}{\gamma}\omega_1} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix},$$

from which together with (24) we have

$$\begin{aligned}b_n=\sum_{k=1}^\infty &\frac{(\int_0^\infty e^{-\gamma x}s(x)dx\int_0^\infty e^{-\gamma x}r(x)dx)^{k-1}}{(1-\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)dx)^k}\\ &\times\{\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_{n+k-1}(\tau)e^{\gamma\tau}d\tau dx\\ &+\int_0^\infty r(x)e^{-\gamma x}dx\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_{n+k}(\tau)e^{\gamma\tau}d\tau dx\\ &+\frac{\lambda}{\gamma}\int_0^\infty y_{n+k-1}(\tau)d\tau+\int_0^\infty r(x)e^{-\gamma x}\int_0^x y_{n+k}(\tau)e^{\gamma\tau}d\tau dx\},\quad n\geq 1.\end{aligned}\tag{25}$$

By (25) and (20) we derive

$$\begin{aligned}a_n=\int_0^\infty e^{-\gamma x}s(x)dx\sum_{k=1}^\infty &\frac{(\int_0^\infty e^{-\gamma x}s(x)dx\int_0^\infty e^{-\gamma x}r(x)dx)^{k-1}}{(1-\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)dx)^k}\\ &\times\{\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_{n+k-1}(\tau)e^{\gamma\tau}d\tau dx\end{aligned}$$

$$+\int_0^\infty r(x)e^{-\gamma x}dx\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_{n+k}(\tau)e^{\gamma\tau}d\tau dx$$
$$+\frac{\lambda}{\gamma}\int_0^\infty y_{n+k-1}(x)dx+\int_0^\infty r(x)e^{-\gamma x}\int_0^x y_{n+k}(\tau)e^{\gamma\tau}d\tau dx\}$$
$$+\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_n(\tau)e^{\gamma\tau}d\tau dx,\quad n\geq 1. \quad (26)$$

Combining (26) with (21) it follows that

$$b_0=\int_0^\infty e^{-\gamma x}r(x)dx\int_0^\infty e^{-\gamma x}s(x)dx$$
$$\times\sum_{k=1}^\infty\frac{(\int_0^\infty e^{-\gamma x}s(x)dx\int_0^\infty e^{-\gamma x}r(x)dx)^{k-1}}{(1-\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)dx)^k}$$
$$\times\{\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_k(\tau)e^{\gamma\tau}d\tau dx$$
$$+\int_0^\infty r(x)e^{-\gamma x}dx\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_{k+1}(\tau)e^{\gamma\tau}d\tau dx$$
$$+\frac{\lambda}{\gamma}\int_0^\infty y_k(\tau)d\tau+\int_0^\infty r(x)e^{-\gamma x}\int_0^x y_{k+1}(\tau)e^{\gamma\tau}d\tau dx\}$$
$$+\int_0^\infty e^{-\gamma x}r(x)dx\int_0^\infty e^{-\gamma x}s(x)\int_0^x z_1(\tau)e^{\gamma\tau}d\tau dx$$
$$+\int_0^\infty e^{-\gamma x}r(x)\int_0^x y_1(\tau)e^{\gamma\tau}d\tau dx+\frac{\lambda}{\gamma+\lambda}y_0. \quad (27)$$

Thus from (26) and changing order of the integrals, we can estimate that (without loss of generality assume $\gamma^2>\alpha(\lambda+\mu)$)

$$\sum_{n=1}^\infty|a_n|\leq\frac{\alpha}{\gamma}\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\lambda\alpha}{\gamma^2})^k}\frac{\alpha\lambda}{\gamma^2}\|z_{n+k-1}\|_{L^1[0,\infty)}$$
$$+\frac{\alpha}{\gamma}\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{(\frac{\mu\alpha}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\alpha\mu}{\gamma^2}\|z_{n+k}\|_{L^1[0,\infty)}$$
$$+\frac{\alpha}{\gamma}\sum_{n=1}^\infty\|z_n\|_{L^1[0,\infty)}+\frac{\alpha}{\gamma}\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{(\frac{\mu\alpha}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\lambda}{\gamma}\|y_{n+k-1}\|_{L^1[0,\infty)}$$
$$+\frac{\alpha}{\gamma}\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\mu}{\gamma}\|y_{n+k}\|_{L^1[0,\infty)}$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\lambda\alpha}{\gamma^2})^k}\frac{\alpha^2\lambda}{\gamma^3}\|z_n\|_{L^1[0,\infty)} \\
&+\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{(\frac{\mu\alpha}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\alpha^2\mu}{\gamma^3}\|z_n\|_{L^1[0,\infty)}+\frac{\alpha}{\gamma}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(\frac{\mu\alpha}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\lambda\alpha}{\gamma^2}\|y_n\|_{L^1[0,\infty)} \\
&+\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\mu\alpha}{\gamma^2}\|y_n\|_{L^1[0,\infty)}=\{\sum_{k=1}^{\infty}\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\frac{\alpha^2\lambda}{\gamma^3} \\
&+\frac{\alpha}{\gamma}\}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)}+\frac{\alpha^2\mu}{\gamma^3}\sum_{k=1}^{\infty}\frac{(\frac{\mu\alpha}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\sum_{n=2}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\frac{\alpha\lambda}{\gamma^2}\sum_{k=1}^{\infty}\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{\alpha\mu}{\gamma^2}\sum_{k=1}^{\infty}\frac{(\frac{\mu\alpha}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\sum_{n=2}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&=\{\frac{1}{\gamma^2-\alpha\lambda-\alpha\mu}\frac{\alpha^2\lambda}{\gamma}+\frac{\alpha}{\gamma}\}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\frac{\alpha^2\mu}{\gamma}\frac{1}{\gamma^2-\alpha\lambda-\alpha\mu}\sum_{n=2}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\frac{\alpha\lambda}{\gamma^2-\alpha\lambda-\alpha\mu}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{\alpha\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\sum_{n=2}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&\leq\frac{\gamma\alpha}{\gamma^2-\alpha\lambda-\alpha\mu}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)}+\frac{\alpha(\lambda+\mu)}{\gamma^2-\alpha\lambda-\alpha\mu}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}. \qquad (28)
\end{aligned}
$$

In (28) we used the following inequalities

$$
\int_0^{\infty}e^{-\gamma x}s(x)dx\leq\alpha\int_0^{\infty}e^{-\gamma x}dx=\frac{\alpha}{\gamma}
\Longrightarrow\frac{1}{1-\frac{\lambda}{\gamma}\int_0^{\infty}e^{-\gamma x}s(x)dx}\leq\frac{1}{1-\frac{\lambda\alpha}{\gamma^2}}
$$

and

$$
\sum_{n=2}^{\infty}\|y_n\|_{L^1[0,\infty)}\leq\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)},\quad\sum_{n=2}^{\infty}\|z_n\|_{L^1[0,\infty)}\leq\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)}.
$$

By (27) and (25) we estimate that

$$
\begin{aligned}
\sum_{n=0}^{\infty}|b_n| &= |b_0| + \sum_{n=1}^{\infty}|b_n| \\
&\leq \sum_{k=1}^{\infty}\Big(\frac{\frac{\alpha\mu}{\gamma^2}}{1-\frac{\lambda\alpha}{\gamma^2}}\Big)^k\Big\{\frac{\alpha\lambda}{\gamma^2}\|z_k\|_{L^1[0,\infty)} + \frac{\alpha\mu}{\gamma^2}\|z_{k+1}\|_{L^1[0,\infty)} \\
&\qquad + \frac{\lambda}{\gamma}\|y_k\|_{L^1[0,\infty)} + \frac{\mu}{\gamma}\|y_{k+1}\|_{L^1[0,\infty)}\Big\} \\
&\quad + \frac{\alpha\mu}{\gamma^2}\|z_1\|_{L^1[0,\infty)} + \frac{\mu}{\gamma}\|y_1\|_{L^1[0,\infty)} + \frac{\lambda}{\gamma+\lambda}|y_0| \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(\frac{\alpha\mu}{\gamma^2})^{k-1}}{(1-\frac{\alpha\lambda}{\gamma^2})^k}\Big\{\frac{\alpha\lambda}{\gamma^2}\|z_{n+k-1}\|_{L^1[0,\infty)} \\
&+ \frac{\alpha\mu}{\gamma^2}\|z_{n+k}\|_{L^1[0,\infty)} + \frac{\lambda}{\gamma}\|y_{n+k-1}\|_{L^1[0,\infty)} + \frac{\mu}{\gamma}\|y_{n+k}\|_{L^1[0,\infty)}\Big\} \\
&\leq \frac{\frac{\alpha\mu}{\gamma^2}}{1-\frac{\lambda\alpha}{\gamma^2}}\frac{\lambda\alpha}{\gamma^2}\sum_{k=1}^{\infty}\|z_k\|_{L^1[0,\infty)} + \frac{\frac{\alpha\mu}{\gamma^2}}{1-\frac{\lambda\alpha}{\gamma^2}}\frac{\alpha\mu}{\gamma^2}\sum_{k=1}^{\infty}\|z_{k+1}\|_{L^1[0,\infty)} \\
&\quad + \frac{\lambda}{\gamma}\frac{\frac{\alpha\mu}{\gamma^2}}{1-\frac{\alpha\lambda}{\gamma^2}}\sum_{k=1}^{\infty}\|y_k\|_{L^1[0,\infty)} + \frac{\mu}{\gamma}\frac{\frac{\alpha\mu}{\gamma^2}}{1-\frac{\alpha\lambda}{\gamma^2}}\sum_{k=1}^{\infty}\|y_{k+1}\|_{L^1[0,\infty)} \\
&\quad + \frac{\alpha\mu}{\gamma^2}\|z_1\|_{L^1[0,\infty)} + \frac{\mu}{\gamma}\|y_1\|_{L^1[0,\infty)} + \frac{\lambda}{\gamma+\lambda}|y_0| \\
&\quad + \frac{\frac{1}{1-\frac{\lambda\alpha}{\gamma^2}}}{1-\frac{\frac{\alpha\mu}{\gamma^2}}{1-\frac{\alpha\lambda}{\gamma^2}}}\sum_{n=1}^{\infty}\Big\{\frac{\lambda\alpha}{\gamma^2}\|z_n\|_{L^1[0,\infty)} + \frac{\alpha\mu}{\gamma^2}\|z_{n+1}\|_{L^1[0,\infty)} \\
&\qquad + \frac{\lambda}{\gamma}\|y_n\|_{L^1[0,\infty)} + \frac{\mu}{\gamma}\|y_{n+1}\|_{L^1[0,\infty)}\Big\} \\
&= \Big\{\frac{\lambda\alpha^2\mu}{\gamma^2(\gamma^2-\lambda\alpha)} + \frac{\alpha\lambda}{\gamma^2-\lambda\alpha-\alpha\mu}\Big\}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&\quad + \Big\{\frac{\alpha^2\mu^2}{\gamma^2(\gamma^2-\lambda\alpha)} + \frac{\alpha\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\Big\}\sum_{n=1}^{\infty}\|z_{n+1}\|_{L^1[0,\infty)} \\
&\quad + \Big\{\frac{\lambda\alpha\mu}{\gamma(\gamma^2-\alpha\lambda)} + \frac{\gamma\lambda}{\gamma^2-\alpha\lambda-\alpha\mu}\Big\}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&\quad + \Big\{\frac{\alpha\mu^2}{\gamma(\gamma^2-\alpha\lambda)} + \frac{\gamma\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\Big\}\sum_{n=1}^{\infty}\|y_{n+1}\|_{L^1[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha\mu}{\gamma^2}\|z_1\|_{L^1[0,\infty)}+\frac{\mu}{\gamma}\|y_1\|_{L^1[0,\infty)}+\frac{\lambda}{\gamma+\lambda}|y_0| \\
&\le\left\{\frac{\alpha^2\lambda\mu}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{\alpha\lambda}{\gamma^2-\alpha\lambda-\alpha\mu}\right. \\
&\left.+\frac{\alpha^2\mu^2}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{\alpha\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\right\}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\left\{\frac{\alpha\lambda\mu}{\gamma(\gamma^2-\alpha\lambda)}+\frac{\gamma\lambda}{\gamma^2-\alpha\lambda-\alpha\mu}+\frac{\alpha\mu^2}{\gamma(\gamma^2-\alpha\lambda)}+\frac{\gamma\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\right\} \\
&\times\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{\lambda}{\gamma+\lambda}|y_0| \\
&=\left\{\frac{\alpha^2\lambda\mu+\alpha^2\mu^2}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{\alpha\lambda+\alpha\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\right\}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\left\{\frac{\alpha\lambda\mu+\alpha\mu^2}{\gamma(\gamma^2-\alpha\lambda)}+\frac{\gamma\lambda+\gamma\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\right\}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{\lambda}{\gamma+\lambda}|y_0|. \qquad (29)
\end{aligned}
$$

In (29) we used the following inequalities

$$
\frac{\alpha^2\mu^2}{\gamma^2(\gamma^2-\lambda\alpha)}+\frac{\alpha\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\ge\frac{\alpha\mu}{\gamma^2}
$$

and

$$
\frac{\alpha\mu^2}{\gamma(\gamma^2-\alpha\lambda)}+\frac{\gamma\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\ge\frac{\mu}{\gamma}.
$$

From (28), (29), (17), (18) and (19) we deduce that

$$
\begin{aligned}
&\|(p,Q)\|=\|p\|+\|Q\|=|p_0|+\sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)}+\sum_{n=0}^{\infty}\|Q_n\|_{L^1[0,\infty)}\le \\
&\frac{1}{\gamma+\lambda}|y_0|+\frac{1}{\gamma}\sum_{n=1}^{\infty}|a_n|+\frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{1}{\gamma}\sum_{n=0}^{\infty}|b_n|+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&\le\frac{1}{\gamma+\lambda}|y_0|+\frac{\alpha}{\gamma^2-\alpha\lambda-\alpha\mu}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\frac{\alpha(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}+\frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&+\{\frac{\alpha^2\mu(\lambda+\mu)}{\gamma^3(\gamma^2-\alpha\lambda)}+\frac{\alpha(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}\}\sum_{n=1}^{\infty}\|z_n\|_{L^1[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\frac{\alpha\mu(\lambda+\mu)}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{\lambda+\mu}{\gamma^2-\alpha\lambda-\alpha\mu}\right\}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&+\frac{\lambda}{\gamma(\gamma+\lambda)}|y_0|+\frac{1}{\gamma}\sum_{n=0}^{\infty}\|z_n\|_{L^1[0,\infty)} \le \left\{\frac{1}{\gamma+\lambda}+\frac{\lambda}{\gamma(\gamma+\lambda)}\right\}|y_0| \\
&+\left\{\frac{\alpha}{\gamma^2-\alpha\lambda-\alpha\mu}+\frac{\alpha^2\mu(\lambda+\mu)}{\gamma^3(\gamma^2-\alpha\lambda)}+\frac{\alpha(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{1}{\gamma}\right\} \\
&\qquad\times\sum_{n=0}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&+\left\{\frac{\alpha(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{\alpha\mu(\lambda+\mu)}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{\lambda+\mu}{\gamma^2-\alpha\lambda-\alpha\mu}+\frac{1}{\gamma}\right\} \\
&\qquad\times\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&=\frac{1}{\gamma}|y_0|+\left\{\frac{\gamma\alpha+\alpha(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{\alpha^2\mu(\lambda+\mu)}{\gamma^3(\gamma^2-\alpha\lambda)}+\frac{1}{\gamma}\right\}\|z\| \\
&+\left\{\frac{\alpha(\lambda+\mu)+\gamma(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{\alpha\mu(\lambda+\mu)}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{1}{\gamma}\right\}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&\le\left\{\frac{\gamma\alpha+\alpha(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{\alpha^2\mu(\lambda+\mu)}{\gamma^3(\gamma^2-\alpha\lambda)}+\frac{1}{\gamma}\right\}\|z\| \\
&+\left\{\frac{\alpha(\lambda+\mu)+\gamma(\lambda+\mu)}{\gamma(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{\alpha\mu(\lambda+\mu)}{\gamma^2(\gamma^2-\alpha\lambda)}+\frac{1}{\gamma}\right\}\|y\| \\
&=\left\{\frac{\gamma^2\alpha(\gamma+\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha^2\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu)}{\gamma^3(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{1}{\gamma}\right\}\|z\| \\
&+\left\{\frac{\gamma(\gamma+\alpha)(\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu)}{\gamma^2(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)}+\frac{1}{\gamma}\right\}\|y\|. \quad (30)
\end{aligned}
$$

If $\alpha \ge \lambda+\mu$, then by a short argument we can verify that the following inequality holds

$$
\begin{aligned}
&\gamma^2\alpha(\gamma^2-\alpha\lambda)(\gamma+\lambda+\mu)+\alpha^2\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu) \\
&\qquad\ge\gamma^2(\gamma+\alpha)(\lambda+\mu)(\gamma^2-\alpha\lambda)+\gamma\alpha\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu) \quad (31)
\end{aligned}
$$

when

$$\gamma^2 \ge \max\left\{\frac{\alpha(\alpha\lambda-\lambda^2+\mu^2)}{\alpha-\lambda-\mu},\alpha(\lambda+\mu)\right\}.$$

(31) allows us to simplify (30) as

$$\|(p,Q)\| \le \left\{ \frac{\gamma^2\alpha(\gamma+\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha^2\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu)}{\gamma^3(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)} + \frac{1}{\gamma} \right\} (\|z\|+\|y\|) = \frac{\gamma^2\alpha(\gamma+\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha^2\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu)+\gamma^2(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)}{\gamma^3(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)} \times \|(y,z)\|. \quad (32)$$

We guarantee that the following two relations are equivalent.

$$-\gamma^4(\lambda+\mu)(\gamma^2-\alpha\lambda)-\alpha^2\gamma^3\{\gamma^2-(\mu^2+\lambda\mu+\lambda\alpha)\} - (\lambda+\mu)\alpha^2\mu\{\gamma^2(\lambda+\mu+\alpha)-\gamma\alpha(\lambda+\mu)-\alpha(\lambda+\mu)(\lambda+\mu+\alpha)\} \le 0$$

$\Longleftrightarrow$

$$\{\gamma-(\lambda+\mu+\alpha)\}\{\gamma^2\alpha(\gamma+\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha^2\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu) + \gamma^2(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)\} \le \gamma^3(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)\,,$$

when

$$\gamma \ge \max\left\{\sqrt{\alpha\lambda}, \sqrt{\mu^2+\alpha\lambda+\lambda\mu}, \frac{\alpha(\lambda+\mu)+\sqrt{\alpha^2(\lambda+\mu)^2+4\alpha(\lambda+\mu)(\lambda+\mu+\alpha)^2}}{2(\lambda+\mu+\alpha)}\right\}.$$

From which together with (32) we can derive that

$$\|(p,Q)\| \le \frac{1}{\gamma-(\lambda+\mu+\alpha)}\|(y,z)\| \quad (33)$$

holds when $\alpha \ge \lambda+\mu$ and

$$\gamma \ge \max\left\{\sqrt{\alpha\lambda}, \sqrt{\mu^2+\alpha\lambda+\lambda\mu}, \frac{\alpha(\lambda+\mu)+\sqrt{\alpha^2(\lambda+\mu)^2+4\alpha(\lambda+\mu)(\lambda+\mu+\alpha)^2}}{2(\lambda+\mu+\alpha)}\right\}.$$

Similarly, if $\alpha < \lambda+\mu$, then the following inequality holds

$$\gamma^2\alpha(\gamma^2-\alpha\lambda)(\gamma+\lambda+\mu)+\alpha^2\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu) \le \gamma^2(\gamma+\alpha)(\lambda+\mu)(\gamma^2-\alpha\lambda)+\gamma\alpha\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu) \quad (34)$$

when

$$\gamma \geq \max\left\{\sqrt{\frac{\alpha\lambda^2}{\lambda+\mu-\alpha}},\right.$$
$$\left.\frac{\alpha\mu(\lambda+\mu)+\sqrt{(\alpha\mu)^2(\lambda+\mu)^2+4\alpha\mu(\lambda+\mu)^2(\alpha\lambda+\mu)^2}}{2(\alpha\lambda+\mu^2)}\right\}.$$

(34) allows us to simplify (30) as

$$\|(p,Q)\| \leq \frac{\gamma(\gamma+\alpha)(\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu)+\gamma(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)}{\gamma^2(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)} \times \|(y,z)\|. \quad (35)$$

By a tedious calculation it can be proved the following equivalent relation holds.

$$-\gamma^3\alpha\{\gamma^2-\alpha\lambda-\mu(\lambda+\mu)\}-\gamma^2(\lambda+\mu)\{(\lambda+\mu)\gamma^2-\alpha\lambda(\lambda+\mu)+\alpha\mu(\lambda+\mu+\alpha)\}$$
$$-\alpha^2\mu(\lambda+\mu)^2\{\gamma-(\lambda+\mu+\alpha)\} \leq 0$$

$$\Longleftrightarrow$$

$$\{\gamma-(\lambda+\mu+\alpha)\}\{\gamma(\gamma+\alpha)(\lambda+\mu)(\gamma^2-\alpha\lambda)+\alpha\mu(\lambda+\mu)(\gamma^2-\alpha\lambda-\alpha\mu)$$
$$+\gamma(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu)\} \leq \gamma^2(\gamma^2-\alpha\lambda)(\gamma^2-\alpha\lambda-\alpha\mu) \quad (36)$$

when

$$\gamma \geq \max\left\{\sqrt{\frac{\alpha\lambda^2}{\lambda+\mu-\alpha}},\right.$$
$$\left.\frac{\alpha\mu(\lambda+\mu)+\sqrt{(\alpha\mu)^2(\lambda+\mu)^2+4\alpha\mu(\lambda+\mu)^2(\alpha\lambda+\mu)^2}}{2(\alpha\lambda+\mu^2)}\right\}.$$

If $\alpha < \lambda+\mu$, then combining (36) with (35) we know that

$$\|(p,Q)\| < \frac{1}{\gamma-(\lambda+\mu+\alpha)}\|(y,z)\|. \quad (37)$$

(33) and (37) show that, when $\gamma \geq \lambda+\mu+\alpha$,

$$(\gamma I-A)^{-1}: X \to X, \quad \|(\gamma I-A)^{-1}\| \leq \frac{1}{\gamma-(\lambda+\mu+\alpha)}, \quad (38)$$

for all α, λ, μ.

As far as the second step is concerned, from $|p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} + \sum_{n=0}^{\infty} \|Q_n\|_{L^1[0,\infty)} < \infty$ for $(p, Q) \in X$, it follows that for any $\epsilon > 0$ there is a positive integer N such that $\sum_{n=N}^{\infty} \|p_n\|_{L^1[0,\infty)} < \epsilon$ and $\sum_{n=N}^{\infty} \|Q_n\|_{L^1[0,\infty)} < \epsilon$. Let

$$L = \Big\{(\overline{p}, \overline{Q}) \big| \overline{p}(x) = (p_0, p_1(x), p_2(x), \cdots, p_N(x), 0, 0, \cdots), p_n \in L^1[0,\infty), \\ n = 1, 2, \cdots, N;\ \overline{Q}(x) = (Q_0(x), Q_1(x), Q_2(x), \cdots, Q_N(x), 0, 0, \cdots), \\ Q_n \in L^1[0,\infty),\ n = 0, 1, 2, \cdots N;\ N \text{ is a finite positive integer}\Big\}.$$

Here $(\overline{p}, \overline{Q}) \in L$ means that the prior finite components of $\overline{p}$ and $\overline{Q}$ are nonzero, while the others are equal to zero. It is obvious that L is dense in X. If we set

$$\mathcal{E} = \Big\{(\overline{p}, \overline{Q})\ \big|\overline{p}(x) = (p_0, p_1(x), p_2(x), \cdots, p_m(x), 0, 0, \cdots), \\ p_i \in C_0^{\infty}[0,\infty), \overline{Q}(x) = (Q_0(x), Q_1(x), Q_2(x), \cdots, Q_m(x), 0, 0, \cdots), \\ Q_j \in C_0^{\infty}[0,\infty), \text{there are positive numbers } c_i > 0\ , d_j > 0 \text{ such that} \\ p_i(x) = 0,\ x \in [0, c_i],\ i = 1, 2, \cdots, m;\ Q_j(x) = 0,\ x \in [0, d_j], \\ j = 0, 1, 2, \cdots, m\Big\}$$

then by [6] it is not difficult to show that $\mathcal{E}$ is dense in L.

From the above discussion we know that, in order to prove that $D(A)$ is dense in X, it suffices to prove that $D(A)$ is dense in $\mathcal{E}$.

Take $(\overline{p}, \overline{Q}) \in \mathcal{E}$, then there exist positive numbers c_i $(i = 1, 2, 3, \cdots, N)$ and d_j $(j = 0, 1, 2, \cdots, N)$ such that

$$\overline{p}(x) = (p_0, p_1(x), p_2(x), \cdots, p_N(x), 0, 0, \cdots), \\ p_i(x) = 0,\ x \in [0, c_i],\ i = 1, 2, \cdots, N;$$

$$\overline{Q}(x) = (Q_0(x), Q_1(x), \cdots, Q_N(x), 0, 0, \cdots), \\ Q_n(x) = 0,\ x \in [0, d_n],\ n = 0, 1, 2, \cdots, N.$$

This yields to

$$p_i(x) = 0, \quad x \in [0, 2\beta], \quad i = 1, 2, \cdots, N;$$

$$Q_j(x) = 0, \quad x \in [0, 2\beta], \quad j = 0, 1, 2, \cdots, N,$$

where $0 < 2\beta < \min\{c_1, c_2, \cdots, c_N, d_0, d_1, d_2, \cdots, d_N\}$. Take

$$F^\beta(0) = (f^\beta(0), g^\beta(0)),$$

$$\begin{aligned} f^\beta(0) &= (p_0, f_1^\beta(0), f_2^\beta(0), f_3^\beta(0), \cdots, f_N^\beta(0), 0, 0, \cdots), \\ &= (p_0, \int_{2\beta}^\infty Q_1(x)s(x)dx, \int_{2\beta}^\infty Q_2(x)s(x)dx, \int_{2\beta}^\infty Q_3(x)s(x)dx, \cdots, \\ &\qquad \int_{2\beta}^\infty Q_N(x)s(x)dx, 0, 0, \cdots), \end{aligned}$$

$$\begin{aligned} g^\beta(0) = (g_0^\beta(0), g_1^\beta(0), g_2^\beta(0), \cdots, g_N^\beta(0), 0, 0, \cdots) \\ = (\lambda p_0 + \int_{2\beta}^\infty p_1(x)r(x)dx, \lambda \int_{2\beta}^\infty p_1(x)dx \\ + \int_{2\beta}^\infty p_2(x)r(x)dx, \\ \lambda \int_{2\beta}^\infty p_2(x)dx + \int_{2\beta}^\infty p_3(x)r(x)dx, \cdots, \lambda \int_{2\beta}^\infty p_{N-1}(x)dx \\ + \int_{2\beta}^\infty p_N(x)r(x)dx, \\ \lambda \int_0^\beta f_N^\beta(0)(1 - \frac{x}{\beta})^2 dx - u_N \lambda \int_\beta^{2\beta} (x-\beta)^2 (x-2\beta)^2 dx \\ + \lambda \int_{2\beta}^\infty p_N(x)dx, 0, 0, \cdots), \end{aligned}$$

$$F^\beta(x) = (f^\beta(x), g^\beta(x)),$$

$$f^\beta(x) = (p_0, f_1^\beta(x), f_2^\beta(x), \cdots, f_N^\beta(x), 0, 0, \cdots),$$

$$g^\beta(x) = (g_0^\beta(x), g_1^\beta(x), g_2^\beta(x), \cdots, g_N^\beta(x), 0, 0, \cdots).$$

Here

$$f_i^\beta(x) = \begin{cases} f_i^\beta(0)(1 - \frac{x}{\beta})^2 & x \in [0, \beta], \\ -u_i(x-\beta)^2(x-2\beta)^2 & x \in [\beta, 2\beta], \\ p_i(x) & x \in [2\beta, \infty), \end{cases} \qquad i = 1, 2, 3, \cdots, N;$$

$$g_j^\beta(x)=\begin{cases} g_j^\beta(0)(1-\frac{x}{\beta})^2 & x\in[0,\beta],\\ -v_j(x-\beta)^2(x-2\beta)^2 & x\in[\beta,2\beta],\\ Q_j(x) & x\in[2\beta,\infty), \end{cases} \qquad j=0,1,2,\cdots,N;$$

$$v_j=\frac{g_i^\beta(0)\int_0^\beta(1-\frac{x}{\beta})^2 s(x)dx}{\int_\beta^{2\beta}(x-\beta)^2(x-2\beta)^2 s(x)dx}, \quad j=1,2,3,\cdots,N;$$

$$u_1=\frac{f_1^\beta(0)\int_0^\beta(1-\frac{x}{\beta})^2 r(x)dx}{\int_\beta^{2\beta}(x-\beta)^2(x-2\beta)^2 r(x)dx},$$

$$u_i=\frac{1}{\int_\beta^{2\beta}(x-\beta)^2(x-2\beta)^2 r(x)dx}\{\lambda\int_0^\beta f_{i-1}^\beta(0)(1-\frac{x}{\beta})^2dx$$
$$-\lambda u_{i-1}\int_\beta^{2\beta}(x-\beta)^2(x-2\beta)^2dx+\int_0^\beta f_i^\beta(0)(1-\frac{x}{\beta})^2 r(x)dx\},$$

where $i=2,3,\cdots,N$. It is easy to verify that $F^\beta\in D(A)$, and moreover

$$\|(\overline{p},\overline{Q})-(f,g)\|=\sum_{n=1}^N\int_0^\infty|p_n(x)-f_n^\beta(x)|dx+\sum_{n=0}^N\int_0^\infty|Q_n(x)-g_n^\beta(x)|dx$$
$$\le\sum_{n=1}^N|f_n^\beta(0)|\frac{\beta}{3}+\sum_{n=1}^N|u_n|\frac{\beta^5}{30}+\sum_{n=0}^N|g_n^\beta(0)|\frac{\beta}{3}+\sum_{n=0}^N|v_n|\frac{\beta^5}{30}\longrightarrow 0,$$
$$\text{as}\quad\beta\to 0.$$

This shows that $D(A)$ is dense in $\mathcal{E}$. In other words, $D(A)$ is dense in X. From the first, second steps and the Hille-Yosida Theorem (see Goldstein [3], Pazy [8]) we deduce that A generates a C_0- semigroup.

In the third step we prove that U and E are bounded linear operators. For any $(p,Q)\in X$ we have

$$\|U(p,Q)\|\le\sum_{n=1}^\infty\int_0^\infty(\lambda+r(x))|p_n(x)|dx+\sum_{n=0}^\infty\int_0^\infty(\lambda+s(x))|Q_n(x)|dx$$
$$+\sum_{n=0}^\infty\lambda\int_0^\infty|Q_n(x)|dx$$
$$\le\sum_{n=1}^\infty(\lambda+\mu)\int_0^\infty|p_n(x)|dx+\sum_{n=0}^\infty(\lambda+\alpha+\lambda)\int_0^\infty|Q_n(x)|dx$$

$$\le (\lambda+\mu)\|p\| + (2\lambda+\alpha)\|Q\| \le (2\lambda+\alpha+\mu)(\|p\|+\|Q\|) = (2\lambda+\alpha+\mu)\|(p,Q)\|. \quad (39)$$

$$\|E(p,Q)\| \le \int_0^\infty |Q_0(x)|s(x)dx \le \alpha \int_0^\infty |Q_0(x)|dx = \alpha\|Q_0\|_{L^1[0,\infty)} \le \alpha\|Q\| \le \alpha\|(p,Q)\|. \quad (40)$$

It is evident that U and E are linear operators, which together with (39) and (40) we know that U and E are bounded linear operators.

From the first, second, third steps and [3, 8] we obtain that $A+U+E$ generates a C_0- semigroup $T(t)$.

In the fourth step we prove that $A+U+E$ is dispersive. For $(p,Q) \in D(A)$ we may choose

$$G(x) = (\phi(x), \psi(x)),$$
$$\phi(x) = (\frac{[p_0]^+}{p_0}, \frac{[p_1(x)]^+}{p_1(x)}, \frac{[p_2(x)]^+}{p_2(x)}, \cdots),$$
$$\psi(x) = (\frac{[Q_0(x)]^+}{Q_0(x)}, \frac{[Q_1(x)]^+}{Q_1(x)}, \frac{[Q_2(x)]^+}{Q_2(x)}, \cdots),$$

here

$$[p_0]^+ = \begin{cases} p_0 & p_0 > 0, \\ 0 & p_0 \le 0, \end{cases}$$
$$[p_n(x)]^+ = \begin{cases} p_n(x) & p_n(x) > 0, \\ 0 & p_n(x) \le 0, \end{cases} \quad n = 1, 2, 3, \cdots ;$$
$$[Q_n(x)]^+ = \begin{cases} Q_n(x) & Q_n(x) > 0, \\ 0 & Q_n(x) \le 0, \end{cases} \quad n = 0, 1, 2, 3, \cdots .$$

If we set $V_n = \{x \in [0,\infty) \mid p_n(x) > 0\}$ and $W_n = \{x \in [0,\infty) \mid p_n(x) \le 0\}$ for $n \ge 1$, then we have

$$\int_0^\infty \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx = \int_{V_n} \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx + \int_{W_n} \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx$$
$$= \int_{V_n} \frac{dp_n(x)}{dx}dx = \int_0^\infty \frac{d[p_n(x)]^+}{dx}dx = -[p_n(0)]^+, \quad n \ge 1. \quad (41)$$

Similarly

$$\int_0^\infty \frac{dQ_n(x)}{dx}\frac{[Q_n(x)]^+}{Q_n(x)}dx = -[Q_n(0)]^+, \quad n \geq 0. \tag{42}$$

From the boundary conditions on (p, Q) we can estimate

$$\sum_{n=1}^{\infty}[p_n(0)]^+ \leq \sum_{n=1}^{\infty}\int_0^\infty [Q_n(x)]^+ s(x)dx\,, \tag{43}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}[Q_n(0)]^+ &= [Q_0(0)]^+ + \sum_{n=1}^{\infty}[Q_n(0)]^+ \\
&\leq \lambda[p_0]^+ + \int_0^\infty [p_1(x)]^+ r(x)dx + \sum_{n=1}^{\infty}\lambda\int_0^\infty [p_n(x)]^+dx \\
&\quad + \sum_{n=1}^{\infty}\int_0^\infty [p_{n+1}(x)]^+ r(x)dx \\
&= \lambda[p_0]^+ + \lambda\sum_{n=1}^{\infty}\int_0^\infty [p_n(x)]^+dx + \sum_{n=1}^{\infty}\int_0^\infty [p_n(x)]^+ r(x)dx \\
&= \lambda[p_0]^+ + \sum_{n=1}^{\infty}\int_0^\infty (\lambda + r(x))[p_n(x)]^+dx.
\end{aligned} \tag{44}$$

By using (41)–(44) we obtain

$$\begin{aligned}
\langle (A+U+E)(p,Q),(\phi,\psi)\rangle &= [-\lambda p_0 + \int_0^\infty Q_0(x)s(x)dx]\frac{[p_0]^+}{p_0} \\
&\quad + \sum_{n=1}^{\infty}\int_0^\infty \{-\frac{dp_n(x)}{dx} - (\lambda + r(x))p_n(x)\}\frac{[p_n(x)]^+}{p_n(x)}dx \\
&\quad + \sum_{n=0}^{\infty}\int_0^\infty \{-\frac{dQ_n(x)}{dx} - (\lambda + s(x))Q_n(x)\}\frac{[Q_n(x)]^+}{Q_n(x)}dx \\
&\quad + \sum_{n=1}^{\infty}\int_0^\infty \lambda Q_{n-1}(x)\frac{[Q_n(x)]^+}{Q_n(x)}dx \\
&\leq -\lambda[p_0]^+ + \frac{[p_0]^+}{p_0}\int_0^\infty Q_0(x)s(x)dx \\
&\quad + \sum_{n=1}^{\infty}\int_0^\infty [Q_n(x)]^+ s(x)dx - \sum_{n=1}^{\infty}\int_0^\infty (\lambda + r(x))[p_n(x)]^+dx
\end{aligned}$$

$$+\lambda[p_0]^+ + \sum_{n=1}^{\infty}\int_0^{\infty}(\lambda + r(x))[p_n(x)]^+ dx$$
$$-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda + s(x))[Q_n(x)]^+ dx + \sum_{n=1}^{\infty}\int_0^{\infty}\lambda Q_{n-1}(x)\frac{[Q_n(x)]^+}{Q_n(x)}dx$$
$$\leq \frac{[p_0]^+}{p_0}\int_0^{\infty}[Q_0(x)]^+ s(x)dx - \int_0^{\infty}[Q_0(x)]^+ s(x)dx$$
$$= (\frac{[p_0]^+}{p_0} - 1)\int_0^{\infty}[Q_0(x)]^+ s(x)dx \leq 0. \quad (45)$$

In (45) we used

$$\int_0^{\infty}\lambda Q_{n-1}(x)\frac{[Q_n(x)]^+}{Q_n(x)}dx \leq \int_0^{\infty}\lambda[Q_{n-1}(x)]^+ dx, \quad n \geq 1.$$

(45) shows that $A+U+E$ is dispersive. By [7, p. 249] we know that $A+U+E$ generates a positive contraction C_0- semigroup. By the uniqueness of C_0- semigroup [8] we conclude that this positive contraction C_0- semigroup is just $T(t)$. The proof of Theorem 1 is complete. □

It is not difficult to prove that X^*, the dual space of X, is as follows

$$X^* = \{(p^*, Q^*) \mid \|(p^*, Q^*)\| = \sup\{\|p^*\|, \|Q^*\|\} < \infty\},$$

here

$$p^* \in R \times L^{\infty}[0,\infty) \times L^{\infty}[0,\infty) \times L^{\infty}[0,\infty) \times \cdots,$$
$$Q^* \in L^{\infty}[0,\infty) \times L^{\infty}[0,\infty) \times L^{\infty}[0,\infty) \times \cdots,$$
$$\|p^*\| = \sup\{|p_0^*|, \sup_{n\geq 1}\|p_n^*\|_{L^{\infty}[0,\infty)}\}, \quad \|Q^*\| = \sup_{n\geq 0}\|Q_n\|_{L^{\infty}[0,\infty)}.$$

It is obvious that X^* is a Banach space. In X we take a set

$$Y = \left\{(p,Q) \in X \;\middle|\; \begin{array}{l} p_0 \geq 0,\ p_n(x) \geq 0,\ Q_i(x) \geq 0,\ n = 1,2,\cdots, \\ i = 0,1,2,\cdots,\ x \in [0,\infty) \end{array}\right\}.$$

Then Theorem 1 ensures that

$$T(t)Y \subset Y. \quad (46)$$

If $(p,Q) \in D(A) \cap Y$, then we choose

$$(p^*, Q^*) = \left(\|(p,Q)\|\begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix}, \|(p,Q)\|\begin{pmatrix}1\\1\\1\\\vdots\end{pmatrix}\right).$$

Thus $(p^*, Q^*) \in X^*$ and moreover

$$\begin{aligned}\langle (p,Q),(p^*,Q^*)\rangle &= \langle p,p^*\rangle + \langle Q,Q^*\rangle \\ &= \|(p,Q)\|p_0 + \sum_{n=1}^{\infty}\|(p,Q)\|\int_0^\infty p_n(x)dx + \sum_{n=0}^{\infty}\|(p,Q)\|\int_0^\infty Q_n(x)dx \\ &= \|(p,Q)\|(p_0 + \sum_{n=1}^{\infty}\int_0^\infty p_n(x)dx) + \|(p,Q)\|\|Q\| \\ &= \|(p,Q)\|\|p\| + \|(p,Q)\|\|Q\| = \|(p,Q)\|\|(p,Q)\| \\ &= \|(p,Q)\|^2 = \|(p^*,Q^*)\|^2 ,\end{aligned}$$

which shows that $(p^*, Q^*) \in \theta((p,Q))$. Here

$$\theta((p,Q)) = \{(p^*,Q^*) \in X^* \mid \langle (p,Q),(p^*,Q^*)\rangle = \|(p,Q)\|^2 = \|(p^*,Q^*)\|^2\}.$$

In addition, we have, for $(p,Q) \in D(A)$ and $(p^*,Q^*) \in \theta((p,Q))$

$$\begin{aligned}\langle (A+U+E)(p,Q),(p^*,Q^*)\rangle &= \|(p,Q)\|\{-\lambda p_0 + \int_0^\infty Q_0(x)s(x)dx\} \\ &+ \sum_{n=1}^{\infty}\int_0^\infty \|(p,Q)\|\{-\frac{dp_n(x)}{dx} - (\lambda + r(x))p_n(x)\}dx \\ &+ \sum_{n=0}^{\infty}\int_0^\infty \|(p,Q)\|\{-\frac{dQ_n(x)}{dx} - (\lambda + s(x))Q_n(x)\}dx \\ &+ \sum_{n=1}^{\infty}\int_0^\infty \|(p,Q)\|\lambda Q_{n-1}(x)dx \\ &= \|(p,Q)\|\{-\lambda p_0 + \int_0^\infty Q_0(x)s(x)dx\} \\ &+ \|(p,Q)\|\sum_{n=1}^{\infty}p_n(0) - \|(p,Q)\|\sum_{n=1}^{\infty}\int_0^\infty(\lambda + r(x))p_n(x)dx \\ &+ \|(p,Q)\|\sum_{n=0}^{\infty}Q_n(0) - \|(p,Q)\|\sum_{n=0}^{\infty}\int_0^\infty(\lambda + s(x))Q_n(x)dx \\ &+ \lambda\|(p,Q)\|\sum_{n=1}^{\infty}\int_0^\infty Q_{n-1}(x)dx\end{aligned}$$

$$
\begin{aligned}
&= \|(p,Q)\|\{-\lambda p_0 + \int_0^\infty Q_0(x)s(x)dx\} \\
&+ \|(p,Q)\| \sum_{n=1}^\infty \int_0^\infty Q_n(x)s(x)dx \\
&- \|(p,Q)\| \sum_{n=1}^\infty \int_{n=1}^\infty (\lambda + r(x))p_n(x)dx \\
&+ \|(p,Q)\|\{\lambda p_0 + \lambda \sum_{n=1}^\infty \int_0^\infty p_n(x)dx \\
&+ \sum_{n=1}^\infty \int_0^\infty p_n(x)r(x)dx\} - \|(p,Q)\| \sum_{n=0}^\infty \int_0^\infty (\lambda + s(x))Q_n(x)dx \\
&+ \lambda\|(p,Q)\| \sum_{n=0}^\infty \int_0^\infty Q_n(x)dx \\
&= \|(p,Q)\| \sum_{n=0}^\infty \int_0^\infty Q_n(x)s(x)dx - \|(p,Q)\| \sum_{n=0}^\infty \int_0^\infty Q_n(x)s(x)dx \\
&= 0. \qquad (47)
\end{aligned}
$$

(47) shows that $A + U + E$ is conservative with respect to $\theta(\cdot)$. Since the initial value of the system (9)–(10) $(p,Q)(0) \in D(A^2) \cap Y$ (see Remark 1), by Fattorini [2, p. 155] we obtain the following result.

Theorem 2. *For the initial value (10),* $p(0) = (1,0,0,\cdots)$, $Q(0) = (0,0,0,\cdots)$, *we have*

$$\|T(t)(p,Q)(0)\| = \|(p,Q)(0)\|, \quad \forall t \in [0,\infty). \qquad (48)$$

Combining Theorem 1 and Theorem 2 with [3, 8] we conclude

Theorem 3. *The system (9)–(10) has a unique positive time-dependent solution* $(p,Q)(x,t)$ *which satisfies*

$$\|(p,Q)(\cdot,t)\| = 1, \quad \forall t \in [0,\infty). \qquad (49)$$

Proof. Since $(p,Q)(0) \in D(A^2) \cap Y$, by [3,8] we know that the system (9) –(10) has a unique positive time-dependent solution $(p,Q)(x,t)$ which can be expressed as

$$(p,Q)(x,t) = T(t)(p,Q)(0). \qquad (50)$$

From this together with (48) we have

$$\begin{aligned}\|(p,Q)(\cdot,t)\| &= \|T(t)(p,Q)(0)\ \| = \|(p,Q)(0)\| \\ &= \|p(0)\| + \|Q(0)\| = 1 + 0 = 1, \quad \forall t \in [0,\infty).\end{aligned}$$

The proof of Theorem 3 is complete. □

(49) just reflects the physical meaning of $(p, Q)(x, t)$.

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