International Journal of Pure and Applied Mathematics

Volume 18 No. 4 2005, 491-493

ON THE NON-DEFECTIVITY AND NON WEAK-DEFECTIVITY OF SEGRE-VERONESE EMBEDDINGS OF PRODUCTS OF $\mathbf{P}^n \times \mathbf{P}^m$

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Fix integers n, m, d, t such that $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1)-n$ and $t \geq mn+1$. Here we prove that the Segre-Veronese embedding of $\mathbf{P}^n \times \mathbf{P}^m$ induced by the complete linear system $|\mathcal{O}_{\mathbf{P}^n \times \mathbf{P}^m}|$ is neither defective nor weakly defective.

AMS Subject Classification: 14N05
Key Words: secant variety, Segre embedding, Veronese embedding, Horace Lemma, zero-dimensional scheme

1. Segre-Veronese Embeddings of $\mathbf{P}^n \times \mathbf{P}^m$

This short note is a continuation of [1]. More precisely, we will make "one more step" in the proof of [1], Theorem 1.1, to obtain the following result.

Theorem 1. *Fix integers n, m, d, t, k such that $k > 0$, $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1) - n$ and $t \geq mn+1$. Let $Z \subset \mathbf{P}^n \times \mathbf{P}^m$ be a general union of k double points. If $k(n+m+1) \leq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbf{P}^n \times \mathbf{P}^m, \mathcal{I}_Z(d,t)) = 0$. If $k(n+m+1) \geq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbf{P}^n \times \mathbf{P}^m, \mathcal{I}_Z(d,t)) = 0$.*

Received: January 5, 2005
© 2005, Academic Publications Ltd.

The characteristic free part of Terracini's Lemma immediately shows that Theorem 1 implies the following result.

Corollary 1. *Fix integers n, m, d, t such that $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1) - n$ and $t \geq mn + 1$. Then the Segre-Veronese embedding of $\mathbf{P}^n \times \mathbf{P}^m$ induced by the complete linear system $|\mathcal{O}_{\mathbf{P}^n \times \mathbf{P}^m}|$ is not defective.*

Using Theorem 1 instead of [1], Theorem 1.1, the proof of [1], Theorem 3, gives verbatim the following result.

Theorem 2. *Fix integers n, m, d, t such that $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1) - n$ and $t \geq mn + 1$. Then the Segre-Veronese embedding of $\mathbf{P}^n \times \mathbf{P}^m$ induced by the complete linear system $|\mathcal{O}_{\mathbf{P}^n \times \mathbf{P}^m}|$ is not weakly defective, i.e. for all integers $k > 0$ such that $k(n + m + 1) < \binom{n+d}{n} \cdot \binom{m+t}{m}$ and any general $S \subset \mathbf{P}^n \times \mathbf{P}^m$ such that $\sharp(S)$ a general hypersurface $F \in |\mathcal{O}_{\mathbf{P}^n \times \mathbf{P}^m}|$ singular at each point of S is smooth outside S and it has an ordinary quadratic singularity at each point of S.*

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. The proof of Theorem 1 and Corollary 1 is characteristic free, while our proof of Theorem 2 heavily depends from the characteristic zero assumption: a key tool is [2], Theorem 1.4. In the proofs of [1], Theorem 1 and Theorem 2, we used an idea of Mella (see [3], proof of Theorem 4.1).

For all integer $n > 0$, $m \geq 0$, $d \geq 0$ and $t \geq 0$ define the integers $a_{(n,m;d,t)}$ and $b_{(n,m;d,t)}$ by the relations:

$$(n + m + 1)a_{(n,m;d,t)} + b_{(n,m;d,t)} = \binom{n+d}{n} \cdot \binom{m+t}{m},$$
$$0 \leq b_{(n,m;d,t)} n + m \,. \quad (1)$$

Proof of Theorem 1. Set $M := \mathbf{P}^n \times \mathbf{P}^m$. Fix a hyperplane $H \subset \mathbf{P}^m$ and set $E := \mathbf{P}^n \times H \cong \mathbf{P}^n \times \mathbf{P}^{m-1}$ (seen as a hypersurface of multidegree $(0, 1)$ of M). By [1], Theorem 1.1, we may assume $m \geq 2$. By induction on m we may also assume that the result is true for $\mathbf{P}^n \times \mathbf{P}^{m-1}$. Subtracting the equation in (1) for the integer $t' := t - 1$ from the same equation for the integer t and using the same equation for the integer $m' := m - 1$ we obtain

$$(n + m + 1)(a_{(n,m;d,t)} - a_{(n,m;d,t-1)}) + b_{(n,m;d,t)} - b_{(n,m;d,t-1)}$$
$$= (n + m)a_{(n,m-1;d,t)} + b_{(n,m-1;d,t)} \,. \quad (2)$$

Since $b_{(n,m;d,t)} \leq n + m$, $b_{(n,m;d,t-1)} \leq n + m$ and $b_{(n,m-1;d,t)} \leq n + m - 1$, from (2) we obtain $a_{(n,m-1;d,x)} \geq (n + m)b_{(n,m-1;d,x)} + a_{(n,m;d,t)} - a_{(n,m;d,t-1)}$. This is the inequality need to make the inductive proof of [1], Theorem 1.1, (from $m' := 0$ to $m'' := m-1$) works in our set-up from the integer $m' := m-1$ to the integer $m'' := m$, using as intermediate step Lemma 1 below instead of Step (a) and Step (b) of the proof of [1], Theorem 1, whose use shows why passing from $m' := m-1$ to $m'' := m-1$ we increase by n our assumed lower bound for t. $\square$

Lemma 1. *Fix integers n, m, d, t, k such that $k > 0$, $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1)-n$ and $t \leq mn$. Let $Z \subset \mathbf{P}^n \times \mathbf{P}^m$ be a general union of k double points. If $(k+nm+1-t)(n+m+1) \leq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbf{P}^n \times \mathbf{P}^m, \mathcal{I}_Z(d,t)) = 0$. If $(k+t-mn-1)(n+m+1) \geq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbf{P}^n \times \mathbf{P}^m, \mathcal{I}_Z(d,t)) = 0$.*

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, On the non-defectivity and non weak-defectivity of Segre-Veronese embeddings of products of projective spaces, *Preprint*.

[2] L. Chiantini, C. Ciliberto, Weakly defective varieties, *Trans. Amer. Math. Soc.*, **454**, No. 1 (2002), 151-178.

[3] M. Mella, Singularities of linear systems and the Waring problem, E-print arXiv mathAG/0406288.