ON THE NON-DEFECTIVITY AND
NON WEAK-DEFECTIVITY OF SEGRE-VERONESE
EMBEDDINGS OF PRODUCTS OF $\mathbb{P}^n \times \mathbb{P}^m$

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Abstract: Fix integers $n, m, d, t$ such that $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1) - n$ and $t \geq mn + 1$. Here we prove that the Segre-Veronese embedding of $\mathbb{P}^n \times \mathbb{P}^m$ induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}|$ is neither defective nor weakly defective.

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1. Segre-Veronese Embeddings of $\mathbb{P}^n \times \mathbb{P}^m$

This short note is a continuation of [1]. More precisely, we will make “one more step” in the proof of [1], Theorem 1.1, to obtain the following result.

Theorem 1. Fix integers $n, m, d, t, k$ such that $k > 0$, $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1) - n$ and $t \geq mn + 1$. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be a general union of $k$ double points. If $k(n + m + 1) \leq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{I}_Z(d, t)) = 0$. If $k(n + m + 1) \geq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{I}_Z(d, t)) = 0$. 
The characteristic free part of Terracini’s Lemma immediately shows that Theorem 1 implies the following result.

**Corollary 1.** Fix integers \( n, m, d, t \) such that \( n \geq m \geq 1, n \geq 2, d \geq n!(n+1) - n \) and \( t \geq mn + 1 \). Then the Segre-Veronese embedding of \( \mathbb{P}^n \times \mathbb{P}^m \) induced by the complete linear system \( |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}| \) is not defective.

Using Theorem 1 instead of [1], Theorem 1.1, the proof of [1], Theorem 3, gives verbatim the following result.

**Theorem 2.** Fix integers \( n, m, d, t \) such that \( n \geq m \geq 1, n \geq 2, d \geq n!(n+1) - n \) and \( t \geq mn + 1 \). Then the Segre-Veronese embedding of \( \mathbb{P}^n \times \mathbb{P}^m \) induced by the complete linear system \( |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}| \) is not weakly defective, i.e. for all integers \( k > 0 \) such that \( k(n + m + 1) < \binom{n+d}{n} \cdot \binom{m+t}{m} \) and any general \( S \subset \mathbb{P}^n \times \mathbb{P}^m \) such that \( \sharp(S) \) a general hypersurface \( F \in |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}| \) singular at each point of \( S \) is smooth outside \( S \) and it has an ordinary quadratic singularity at each point of \( S \).

We work over an algebraically closed field \( \mathbb{K} \) with \( \text{char}(\mathbb{K}) = 0 \). The proof of Theorem 1 and Corollary 1 is characteristic free, while our proof of Theorem 2 heavily depends from the characteristic zero assumption: a key tool is [2], Theorem 1.4. In the proofs of [1], Theorem 1 and Theorem 2, we used an idea of Mella (see [3], proof of Theorem 4.1).

For all integer \( n > 0, m \geq 0, d \geq 0 \) and \( t \geq 0 \) define the integers \( a_{n,m,d,t} \) and \( b_{n,m,d,t} \) by the relations:

\[
(n + m + 1)a_{n,m,d,t} + b_{n,m,d,t} = \binom{n+d}{n} \cdot \binom{m+t}{m},
\]

\[0 \leq b_{n,m,d,t}n + m. \quad (1)\]

**Proof of Theorem 1.** Set \( M := \mathbb{P}^n \times \mathbb{P}^m \). Fix a hyperplane \( H \subset \mathbb{P}^m \) and set \( E := \mathbb{P}^n \times H \cong \mathbb{P}^n \times \mathbb{P}^{m-1} \) (seen as a hypersurface of multidegree \((0,1)\) of \( M \)). By [1], Theorem 1.1, we may assume \( m \geq 2 \). By induction on \( m \) we may also assume that the result is true for \( \mathbb{P}^n \times \mathbb{P}^{m-1} \). Subtracting the equation in (1) for the integer \( t' := t - 1 \) from the same equation for the integer \( t \) and using the same equation for the integer \( m' := m - 1 \) we obtain

\[
(n + m + 1)(a_{n,m,d,t} - a_{n,m,d,t-1}) + b_{n,m,d,t} - b_{n,m,d,t-1} = (n + m)a_{n,m-1,d,t} + b_{n,m-1,d,t}. \quad (2)
\]

Since \( b_{n,m,d,t} \leq n + m \), \( b_{n,m,d,t-1} \leq n + m \) and \( b_{n,m-1,d,t} \leq n + m - 1 \), from (2) we obtain \( a_{n,m-1,d,t} \geq (n + m)b_{n,m-1,d,t} + a_{n,m,d,t} - a_{n,m,d,t-1} \). This
is the inequality need to make the inductive proof of [1], Theorem 1.1, (from $m' := 0$ to $m'' := m - 1$) works in our set-up from the integer $m' := m - 1$ to the integer $m'' := m$, using as intermediate step Lemma 1 below instead of Step (a) and Step (b) of the proof of [1], Theorem 1, whose use shows why passing from $m' := m - 1$ to $m'' := m - 1$ we increase by $n$ our assumed lower bound for $t$.

\textcolor{red}{\textbf{Lemma 1.}} \ Fix integers $n, m, d, t, k$ such that $k > 0$, $n \geq m \geq 1$, $n \geq 2$, $d \geq n!(n+1) - n$ and $t \leq mn$. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be a general union of $k$ double points. If $(k + nm + 1 - t)(n + m + 1) \leq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{I}_Z(d, t)) = 0$. If $(k + t - mn - 1)(n + m + 1) \geq \binom{n+d}{n} \cdot \binom{m+t}{m}$, then $h^1(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{I}_Z(d, t)) = 0$.

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\textbf{References}

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