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A NOTE ON CONTINUITY IN $M|GI|1|\infty$ QUEUES

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Abstract: We offer an upper bound for the uniform distance between stationary distributions of waiting time in two queueing systems. The inequality found provides the uniform bound with respect to the traffic intensity.

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Let us consider two $M|GI|1|\infty$ queues Q and $\tilde{Q}$ with the same Poisson input flow of parameter $\lambda > 0$ and with i.i.d. service times $\xi_1, \xi_2, \dots$ and $\tilde{\xi}_1, \tilde{\xi}_2, \dots$ in Q and $\tilde{Q}$, respectively.

Denoting by ξ and $\tilde{\xi}$ the corresponding generic random variables, and by F_ξ, $F_{\tilde{\xi}}$ their distribution functions we will assume in what follows that:

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$$m := E\xi = E\tilde{\xi}, \tag{1}$$

$$\sigma^2 := \mathrm{Var}(\xi) = \mathrm{Var}(\tilde{\xi}) > 0, \tag{2}$$

$$E|\xi|^3 < \infty,\ E|\tilde{\xi}|^3 < \infty, \tag{3}$$

$$\rho := \lambda m < 1. \tag{4}$$

It is well-known that condition (4) guarantees the existence of the stationary waiting times in the queues Q and $\tilde{Q}$, which we denote by W and $\tilde{W}$, respectively. The moment conditions (1)-(3) allow us to prove the following simple estimate of continuity:

$$\rho_K(W,\tilde{W}) := \sup_{x\geq 0}\left|(P(W\leq x) - P(\tilde{W}\leq x)\right| \leq$$
$$\frac{\rho b}{m}\max\left\{\sup_{x\geq 0}\left|\int_0^x \left[F_\xi(t) - F_{\tilde{\xi}}(t)\right]dt\right|,\ \frac{1}{2}\int_0^\infty t^2\left|F_\xi(t) - F_{\tilde{\xi}}(t)\right|dt\right\}, \tag{5}$$

where

$$b = \max\left\{5, \frac{23.1}{m^2}\left[\frac{4}{3\sigma_*^2} + \frac{1}{\tilde{\sigma}_*^2}\right]\right\},$$

$$\sigma_*^2 = (3m)^{-1}E\xi^3 - (4m^2)^{-1}(\sigma^2+m^2)^2, \tag{6}$$

$$\tilde{\sigma}_*^2 = (3m)^{-1}E\tilde{\xi}^3 - (4m^2)^{-1}(\sigma^2+m^2)^2. \tag{7}$$

There are a number of results on continuity (stability) of the stationary waiting time in queues, particularly, in $M|GI|1|\infty$ queues. Qualitative analysis of continuity can be found in the papers of Kennedy [8], of Whitt [11] and in the book of Stoyan [10]. Quantitative estimates of stability of waiting time distributions are given, for instance, in Kalashnikov and Tsitsiashvili [7], Zolotarev [13], [14], Borovkov [1], Kalashnikov and Rachev [6], Kalashnikov [4]. In Whitt [12] there are problems closely related to continuity in queues.

The approaches used in the above mentioned papers are based on estimating the rate of convergence of waiting time distributions to their stationary version. Consequently these methods do not provide reasonable estimates in the heavy traffic limit $\rho\to 1$. In this note we restrict significantly the class of admissible perturbations of the service time requiring equality of means and variances. As a benefit we obtain estimate (5) which works well for any $\rho\in[0,1]$. Remark that under assumptions (1)-(4) the inequalities in Kalashnikov [4], Chapter 5 give the following bound:

$$\sup_{\rho\in[0,1]}\rho_K(W,\tilde{W}) \leq c\sqrt{\rho_K(\xi,\tilde{\xi})}.$$

In the paper by Gordienko and Ruiz de Chávez [3] we found a bound for the total variation distance between W and $\tilde{W}$ which is uniform in ρ. This bound resembles inequality (5), but its proof appeals to additional assumptions.

To prove (5) we show that this inequality follows from inequalities for distributions of sums of independent random variables in Gordienko [2]. First, we introduce two random variables X and $\tilde{X}$ such that the distribution function of X is

$$F(x) = \frac{1}{m}\int_0^x \big[1 - F_\xi(t)\big]\,dt \tag{8}$$

and the distribution function of $\tilde{X}$ is

$$\tilde{F}(x) = \frac{1}{m}\int_0^x [1 - F_{\tilde{\xi}}(t)]dt. \tag{9}$$

Then (see, e.g. Stoyan [10])

$$\begin{aligned} W(x) &= (1-\rho)\sum_{k=0}^{\infty}\rho^k F^{*k}(x), \\ \tilde{W}(x) &= (1-\rho)\sum_{k=0}^{\infty}\rho^k \tilde{F}^{*k}(x), \end{aligned}$$

where $*$ denotes convolution of distributions.

Applying the total probability formula, we get

$$\rho_K(W,\tilde{W}) \le (1-\rho)\sum_{k=0}^{\infty}\rho^k \rho_K(F^{*k},\tilde{F}^{*k}). \tag{10}$$

It is easy to see from (1), (2), (8) and (9) that $EX_1 = E\tilde{X}_1; EX_1^2, E\tilde{X}_1^2 < \infty$ and

$$\big|\varphi_X(t)\big| \le \frac{2}{m|t|}, \qquad \big|\varphi_{\tilde{X}}(t)\big| \le \frac{2}{m|t|}\,,$$

where φ_X, $\varphi_{\tilde{X}}$ are the corresponding characteristic functions. The last inequalities allow us to apply Theorem 1 in Gordienko [2] that in this case states that

$$\begin{aligned} \rho_K(F^{*k}\tilde{F}^{*k}) \le \max\Bigg\{5, \left[\frac{4}{3\sigma_*^2} + \frac{1}{\tilde{\sigma}_*^2}\right] \\ \times\frac{4}{\pi m^2}\inf_{x\in[0,1]}\left[3x + \frac{4}{x^2(1-x^2)}\right]\Bigg\}\max\Big\{\rho_K(X,\tilde{X}),\ \ \zeta_2(X,\tilde{X}\Big\}, \end{aligned} \tag{11}$$

where $\sigma_*^2 = \mathrm{Var}(X)$, $\tilde{\sigma}_*^2 = \mathrm{Var}(\tilde{X})$ and ζ_2 is Zolotarev's metric of order 2. By virtue of inequality (18.3.19) in Rachev [9]

$$\zeta_2(X,\tilde{X}) \leq \frac{1}{2}\int_0^\infty x^2 \left| dF(x) - d\tilde{F}(x) \right|$$
$$= \frac{1}{2m}\int_0^\infty x^2 \left| F_\xi(x) - F_{\tilde{\xi}}(x) \right| dx. \quad (12)$$

Simple calculations show that $\mathrm{Var}(X)$ and $\mathrm{Var}(\tilde{X})$ are expressed by (6) and (7). Combining (10)-(12) we prove inequality (5).

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