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COVER FOR MODULES, II

N. Amiri[1 §], M. Ershad[2]
[1,2]Department of Mathematics
Shiraz University
Shiraz, 71454, IRAN
[1]e-mail: Amiri@susc.ac.ir

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1. Introduction

Let R be a commutative ring with identity and M be an R-module with $\operatorname{Spec}(M) \neq \phi$. A cover of a submodule K of M is a subset C of $\operatorname{Spec}(M)$ satisfying that for any $x \in K, x \neq 0$, there is $N \in C$, such that $\operatorname{ann}(x) \subset (N : M)$. If we denote by $J = \bigcap_{N \in C} (N : M)$, and assume that M is finitely generated, then $JM = M$, implies that $M = 0$ in [1]. We proved that if R is a Noetherian ring and M is a finitely generated faithful R-module then M has a finite cover and we proved that if R is a Noetherion ring M a finitely generated R-module, C a cover of M and $I \subset \bigcap_{N \in C} (N : M)$, then $\bigcap_{n=1}^{\infty} I^n M = 0$. In this paper we shall see that if C be a subset of $\operatorname{Max}(R)$ then some results are true.

Definition. Let M be a Module over a ring R, a semi cover of M is defined to be a subset C of $\operatorname{Max}(R)$ satisfying that for any $x \in M, x \neq 0$, there is $P \in C$ such that $\operatorname{ann}(x) \subset P$.

Definition. An R-sequence in R is an ordered collection of elements $a_1, a_2, \cdots, a_n \in R$ such that:

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[§]Correspondence author

i) $(a_1, a_2, \cdots, a_n) \neq R$.

ii) $((a_1, a_2, \cdots, a_{i-1}) : a_i) = (a_1, a_2, \cdots, a_{i-1}), 1 \leq i \leq n$.

Definition. Let P be a prime ideal of a ring R. It is said that $\operatorname{ht} P = n$, if there exists a chain

$$P_n \subset P_{n-1} \subset \cdots \subset P_0 = P \text{ of prime ideal } P_i (0 \leq i \leq n)$$

of R, but there is no longer such chain.

Definition. The dimension of a ring $R(\dim R)$ is defined by $\sup\{\operatorname{ht} P : P$ is a prime ideal of $R\}$.

Definition. Let M be an R-module, then we define $\operatorname{Map}(M) = \{x \in M|$ every prime ideal containing $\operatorname{ann}(x)$ is maximal$\}$.

Proposition 1. *Let R be a Noetherian ring, I an ideal of R and M a finitely generated R-module such that $IM \neq M$, then the length of a maximal M-sequence in I is well-determind integer n, and n is determined by* $\operatorname{Ext}^i_R(\frac{R}{I}, M) = 0$ *for $i < n$ and* $\operatorname{Ext}^n_R(\frac{R}{I}, M) \neq 0$ *we write* $\operatorname{dep}_I(M) = n$.

Proof. See [3, Theorem 16.7]. □

Proposition 2. *Let M be an R-module, not Aritinian. Let C be a finite semi cover of M and $J = \bigcap_{P \in C} P$. Then* $\operatorname{Map}(M)$ *is the least element of the set $B = \{N | N$ is a proper submodule of M and* $\operatorname{dep}_J(M/N) > 0\}$.

Proof. Since M is not Artinian, $\operatorname{Map}(M)$ is a proper submodule of M, by [1, Theorem 15] we may assume that $\operatorname{Map}(M) = \operatorname{ann}_M(J^k)$. Now $\operatorname{ann}_{\frac{M}{\operatorname{Map}(M)}}(J) = (\operatorname{Map}(M) : J)/\operatorname{Map}(M) = \frac{\operatorname{ann}_M(J^{k+1})}{\operatorname{ann}_M(J^k)} = 0$. So $\operatorname{dep}_J(\frac{M}{\operatorname{Map}(M)}) > 0$. Hence we have $\operatorname{Map}(M) \in B$. If N be a proper submodule of M satisfying that $\operatorname{dep}_J(M/N) > 0$, so $\operatorname{ann}_{M/N}(J) = \frac{(N:J)}{N} = 0$, then $(N : J) = N$. Hence for any integer $n > 0, (N : J^n) = N$. Thus we get $\operatorname{Map}(M) = \operatorname{ann}(J^k) \subseteq N = (N : J^k)$, so $\operatorname{Map}(M)$ is the least element of B. □

Remark. Let C be a finite semi cover of M and $J = \bigcap_{P \in C} P$. Then $\operatorname{dep}_J(M) > 0$ if and only if $\operatorname{Map}(M) = 0$.

Proposition 3. *Let M be an R-module and $C = \{P_1, P_2, \cdots, P_n\}$ be a finite semi cover of M. Set $J = \bigcap_{i=1}^{n} P_i$, then we have* $\operatorname{Map}(M) = \bigcap_{i=1}^{n} \operatorname{Map}(M_{p_i})^c$, *where $(M_{p_i})^c$ is a contraction of M_{p_i}.*

Proof. Since $\operatorname{Map}(M) \subset (\operatorname{Map}(M)_{P_i})^c \subseteq \operatorname{Map}(M_{P_i})^c$ for all i, we have $\operatorname{Map}(M) \subseteq \bigcap_{i=1}^{n} \operatorname{Map}(M_{P_i})^c$. And from [1, Theorem 15] we can take a fixed integer $k > 0$ such that $\operatorname{Map}(M_{P_i}) = \operatorname{ann}_{M_{P_i}}(P_i^k R_{P_i})$ for all i. Hence $\operatorname{Map}(M_{P_i})^c =$

$\text{ann}_{M_{P_i}}(P_i^k R_{P_i})^c = (\text{ann}_M(P_i^k)_{P_i})^c = \bigcup_{r\in R-P_i}((\text{ann}(P_i^k) :_M r))$. If $x \in \bigcap_{i=1}^{n} \text{Map}(M_{P_i})^c$, then for each i there is $r_i \in R - P_i$ such that $r_i P_i^k x = 0$. Since $r_i R + P_i = R$, we have $P_i^{k+1}x = P_i^k x$. Thus we have $P_1^{k+1}P_2^{k+1}x = P_1^{k=1}P_2^k x = P_2^k P_1^{k+1}x = P_1^k P_2^k x$. And Hence we have $P_1^{k+1}P_2^{k+1}\cdots P_n^{k+1}x = P_1^k P_2^k - P_n^k x$. So $J^{k+1}x = J^k x$ and hence $J^k x = 0$, by [1, Proposition 2]. Thus $x \in \text{ann}_M(J^k) \subset \text{Map}(M)$ and the proof is complete. □

Theorem 4. *Let R be a local ring with the maximal ideal P and M be an R-module. If M is not Aritinian the* $\dim M = \dim(M/\text{Map}(M))$.

Proof. By definition of $\dim(M) = \dim(\frac{R}{\text{ann}(M)})$, and $\dim(\frac{M}{\text{Map}(M)})$, we need to show that rad(ann (M)) =rad(ann $(\frac{M}{\text{Map}(M)})$). This follows from the fact that if $r \in R$ such that $rM \subset \text{Map}(M)$, then $rP^kM \subset P^k\text{Map}(M) = 0$ for some integer $k > 0$, hence $r^{k+1} \in \text{ann}(M)$. □

Proposition 5. *Let C be a finite semi cover of the Notherian ring R, I an ideal of R. If we consider R with the I-adic topology, the following condition are equivalent:*

(1) $I \subset \bigcap_{P\in C} P$.

(2) Zero ideal and every prime ideal contained in $\bigcup_{P\in C} P$ is closed.

(3) If $f : R \longrightarrow \hat{R}$ be the natural map, then $f^{-1}(P\hat{R}) = P$ for every $P \in C$, where $\hat{R}$ is the I-adic completion of R.

Proof. (1) $\Longrightarrow$ (2) Since $\bigcap_{i=1}^{\infty} I^n = 0$, so the zero ideal is closed. If $q \subset \bigcup_{P\in C} P$ is a prime ideal, then $q \subset P$ for some $P \in C$. Since $Ass_R(\frac{R}{P}) = \{P\}$, we see that C is a cover of $\frac{R}{P}$. By [1, Theorem 6], $\bigcap_{n=1}^{\infty}(I^n + P) = P$, hence P is a closed.

(2) $\Longrightarrow$ (3) Since $\{0\}$ is closed, we can assume that $R \subset \hat{R}$. Let $P \in C$, so $P\hat{R}$ is the closure of P in $\hat{R}$ hence $P\hat{R} \cap R$ consists of element of R which are limits of elements contained in P. Since P is closed we have $P\hat{R} \cap R = P$.

(3) $\Longrightarrow$ (1) Since $P\hat{R}$ is closed in $\hat{R}$ and since the map $f : R \longrightarrow \hat{R}$ is continoues, P is closed in R for all $P \in C$. If $I \not\subset \bigcap_{P\in C} P$, then $I \not\subset P$ for some $P \in C$. But then we have $I^k + P = R$ for all $k > 0$, contradicting with the fact that P is closed in I-adic topology. □

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