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CONNES-MODULE-AMENABILITY FOR DUAL MODULES

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Abstract: In this paper, we study the Connes-module-amenability of dual modules. This is a natural generalization of Connes-amenability of dual algebras. For Banach module E, we show that Connes-module-amenability of dual module E'' is equivalent to module amenability E.

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1. Introduction

In [1], the author introduced the concept of module amenability for the class of Banach modules. In this paper we introduce the concept of Connes-module-amenability for a larger class of Banach modules called, dual module. All of the necessary definitions in the following are in [1].

Let $\mathcal{A}$ be a Banach algebra and E be a Banach space with a left $\mathcal{A}$-module structure such that, for some $M > 0$

$$\|a \cdot x\| \leq M\|a\| \, \|x\| \qquad (a \in \mathcal{A}, \ x \in E) \ .$$

Then E is called a left Banach $\mathcal{A}$-module. Right and two-sided Banach $\mathcal{A}$-

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modules are defined similarly. By a Banach $\mathcal{A}$-module, we always mean a two-sided Banach $\mathcal{A}$-module. All over this section E is a left Banach $\mathcal{A}$-module, and $\Delta : E \longrightarrow \mathcal{A}$ is a bounded Banach left $\mathcal{A}$-module homomorphism.

Definition 1.1. Let X be a Banach $\mathcal{A}$-module. A bounded linear map $D : \mathcal{A} \longrightarrow X$ is called a Δ-derivation if

$$D(\Delta(a \cdot x)) = a \cdot D(\Delta(x)) + D(a) \cdot \Delta(x) \qquad (a \in \mathcal{A},\ x \in E)\,.$$

D is called Δ-inner if there is $f \in X'$ such that

$$D(\Delta(x)) = f \cdot \Delta(x) - \Delta(x) \cdot f\ .$$

It is easy to see that, when Δ has a dense range, Δ-derivation extends uniquely to a derivation from $\mathcal{A}$ to X'.

Definition 1.2. E is called Δ-amenable as a left $\mathcal{A}$-module, if for each Banach $\mathcal{A}$-module X, all Δ-derivations from $\mathcal{A}$ to X' are Δ-inner. Right and two-sided Δ-amenability are defined similarly.

It is clear $\mathcal{A}$ is Δ-amenable as a Banach $\mathcal{A}$-module (with Δ =id) iff it is amenable as a Banach algebra.

Amenability in the sense of [3], can be intrinsically characterized in terms of so-called approximate and virtual diagonal, [2]. There is a related notion for module amenability. See [1].

2. Connes-Module-Amenability

Let E be a Banach $\mathcal{A}$-module. Then E', the dual space of E, is a Banach $\mathcal{A}$-module for the operations given by

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \qquad (a \in \mathcal{A},\ x \in E,\ f \in E')\ .$$

Definition 2.1. E is called dual $\mathcal{A}$-module, if there is a closed submodule E_* of E' such that $E = (E_*)'$.

Example. 1.) If G is a locally compact group, then $C_0(G)$ is a Banach $L^1(G)$-module, and $M(G) = C_0(G)'$ is a dual $L^1(G)$-module.

2.) $L^\infty(G)$ is a dual $L^1(G)$-module.

3.) Every dual algebra is a dual module.

All over this section, E is a dual left $\mathcal{A}$-module, and $\Delta : E \longrightarrow \mathcal{A}$ is a bounded Banach left $\mathcal{A}$-module homomorphism.

Proposition 2.2. *For each $a \in \mathcal{A}$ the maps,*

$$E \longrightarrow E, \quad e \longmapsto a \cdot e, \quad e \longmapsto e \cdot a$$

are $w^* - w^*$ continuous.

Proof. It is obvious. □

Definition 2.3. A Banach $\mathcal{A}$-module X, is called Δ-normal if for each $\varphi \in X'$, the maps $E \longrightarrow E, \ e \longmapsto (\Delta e) \cdot \varphi, \ e \longmapsto \varphi \cdot (\Delta e)$ are $w^* - w^*$ continuous.

We can now define Connes-module-amenable, dual modules.

Definition 2.4. E is called left Connes-module-amenable (Connes-Δ-amenable) if for every Δ-normal Banach $\mathcal{A}$-module X, every Δ-derivation, $D : \mathcal{A} \longrightarrow X'$ such that $D \circ \Delta : E \longrightarrow X'$ be $w^* - w^*$ continuous, is Δ-inner. Right and two-sided Connes-module-amenability are defined similarly.

It is clear that dual algebra $\mathcal{A}$ is Connes-Δ-amenable as a dual $\mathcal{A}$-module (with Δ =id) iff it is Connes-amenable as a dual algebra (see [4]).

Proposition 2.5. *Let Δ is a w^*-norm continuous, isomorphism, if dual module E is Connes-Δ-amenable, then $\mathcal{A}$ has an right identity for $\Delta(E)$.*

Proof. Let X be the Banach $\mathcal{A}$-module whose underlying linear space is E_* equipped with the following module operations:

$$a \cdot x = 0 \quad \text{and} \quad x \cdot a = xa \quad (a \in \mathcal{A}, \ x \in E_*) .$$

Obviously, X, is a Δ-normal Banach $\mathcal{A}$-module. By the Han-Banach theorem there is a bounded linear map $D : \mathcal{A} \longrightarrow E$ such that id$= D \circ \Delta : E \longrightarrow E$ is $w^* - w^*$ continuous and

$$D(\Delta(a \cdot x)) = a \cdot D(\Delta(x)) + D(a) \cdot \Delta(x) \qquad (a \in \mathcal{A}, \ x \in E) \ .$$

Since E is Connes-Δ-amenable, there is $e \in E$ such that

$$(D \circ \Delta)(e) = \Delta(e) \cdot e' - e' \cdot (\Delta e) \implies e = (\Delta e) \cdot e' \implies \Delta e = (\Delta e) \cdot (\Delta e') \, ,$$

so that $\Delta(e')$ is a right identity for $\Delta(E)$. □

Proposition 2.6. *Let $\mathcal{A}$ is a dual algebra and E a dual $\mathcal{A}$-module and $\Delta : E \longrightarrow \mathcal{A}$ is a epimorphism. Then $\mathcal{A}$ is Connes-amenable iff E is Connes-Δ-amenable.*

Proof. Follows from definitions and [4]. □

Proposition 2.7. *Let E and E' are dual $\mathcal{A}$-modules and $\Delta : E \longrightarrow \mathcal{A}$ is a w^*-norm continuous module homomorphism and $\Theta : E' \longrightarrow E$ is a $w^* - w^*$*

continuous module homomorphism with w^*-dense range. If E' is Connes-$\Delta\circ\Theta$-amenable, then E is Connes-Δ-amenable.

Proof. Let X is a Δ-normal Banach $\mathcal{A}$-module and $D:\mathcal{A}\longrightarrow X'$ is a bounded linear map such that $D\circ\Delta:E\longrightarrow X'$ is w^*-w^* continuous and

$$D(\Delta(a\cdot x)) = a\cdot D(\Delta(x)) + D(a)\cdot\Delta(x) \qquad (a\in\mathcal{A},\ x\in E)\,,$$

for $f\in X'$, the maps, $E'\longrightarrow X'$, $y\longmapsto(\Delta\circ\Theta)(y)\cdot f$, $y\longmapsto f\cdot(\Delta\circ\Theta)(y)$ is w^*-w^* continuous and

$$D((\Delta\circ\Theta)(a\cdot y)) = D(\Delta(a\cdot\Theta(y)) = a\cdot D((\Delta\circ\Theta)(y)) + D(a)\cdot(\Delta\circ\Theta)(y)$$

since E' is Connes-$\Delta\circ\Theta$-amenable, there is $f\in X'$ such that

$$D((\Delta\circ\Theta)(y)) = (\Delta\circ\Theta)(y)\cdot f - f\cdot(\Delta\circ\Theta)(y) \qquad (y\in E')\,,$$

for $x\in E$ there is a net $\{e_\alpha\}$ in E', such that $\varphi(e_\alpha)\xrightarrow{w^*}x$. Therefore

$$\begin{aligned} D(\Delta(x)) &= \lim_\alpha [D(\Delta\circ\Theta)(e_\alpha)] \\ &= \lim_\alpha(\Delta\circ\Theta)(e_\alpha)\cdot f - \lim_\alpha f\cdot(\Delta\circ\Theta)(e_\alpha) \\ &= (\Delta x)\cdot f - f\cdot(\Delta x). \end{aligned}$$

Proposition 2.8. *Assume that E is a Banach $\mathcal{A}$-module and E' a dual $\mathcal{A}$-module and $\Delta:E\longrightarrow\mathcal{A}$, $\Theta:E'\longrightarrow E$ are module homomorphism such that* $\mathrm{Im}(\Theta)$ *is dense in E. If E is Δ-amenable. Then E' is Connes-$\Delta\circ\Theta$-amenable.*

Proof. Let X is a $\Delta\circ\Theta$-normal Banach $\mathcal{A}$-module and $D:\mathcal{A}\longrightarrow X'$ is a bounded linear map such that $D\circ(\Delta\circ\Theta):E'\longrightarrow X'$ is w^*-w^* continuous, and

$$D((\Delta\circ\Theta)(a\cdot y)) = a\cdot D((\Delta\circ\Theta)(y)) + (Da)\cdot(\Delta\circ\Theta)(y) \qquad (a\in\mathcal{A},\ y\in E')$$

for $x\in E$ there is a net $\{e_\alpha\}$ in E' such that $\varphi(e_\alpha)\longrightarrow x$ (in norm), therefore:

$$\begin{aligned} D(\Delta(a\cdot x)) &= \lim_\alpha D(\Delta(e_\alpha))\cdot(Da) + \lim_\alpha a\cdot D((\Delta\circ\Theta)(e_\alpha)) \\ &= (\Delta(x))\cdot(Da) + a\cdot D(\Delta(x))\ . \end{aligned}$$

Since E is Δ-amenable, there is $f\in X'$ such that $D(\Delta(x)) = (\Delta x)\cdot f - f\cdot(\Delta x)$. Hence

$$D((\Delta\circ\Theta)(y)) = (\Delta\circ\Theta)(y)\cdot f - f\cdot(\Delta\circ\Theta)(y).$$ □

Proposition 2.9. *Assume that E is a Banach $\mathcal{A}$-module and E' a dual $\mathcal{A}$-module and $\Delta : E' \longrightarrow \mathcal{A}$, $\Theta : E' \longrightarrow E$ are module homomorphism such that Im(Θ) is w^*-dense in E'. If E is $\Delta\circ\Theta$-amenable then E' is Connes-Δ-amenable.*

Proof. Let X is a Δ-normal Banach module, and $D : \mathcal{A} \longrightarrow X'$ is a bounded linear map such that, $D \circ \Delta : E \longrightarrow X'$ is $w^* - w^*$ continuous and

$$D(\Delta(a \cdot y)) = a \cdot D(\Delta(y)) + (Da) \cdot (\Delta y) \qquad (a \in \mathcal{A},\ y \in E')$$

for $x \in E$. We have:

$$D((\Delta \circ \Theta)(a \cdot x)) = D(\Delta(a \cdot \Theta(x)) = a \cdot D((\Delta \circ \Theta)(x)) + (Da) \cdot (\Delta \circ \Theta)(x)\,.$$

Since E' is $\Delta \circ \Theta$-amenable, there is $f \in X'$ such that

$$D((\Delta \circ \Theta)(x)) = (\Delta \circ \Theta)(x) \cdot f - f \cdot (\Delta \circ \Theta)(x)\ .$$

For $y \in E'$, there is a net $\{e_\alpha\}$ in E such that $\varphi(e_\alpha) \xrightarrow{w^*} y$. Therefore

$$\begin{aligned} D(\Delta(y)) &= \lim_\alpha \Delta(\Theta(e_\alpha)) \cdot f - \lim_\alpha f \cdot \Delta(\Theta(e_\alpha)) \\ &= (\Delta y) \cdot f - f \cdot (\Delta y). \quad \square \end{aligned}$$

Corollary 2.10. *Let E is a Banach $\mathcal{A}$-module with second dual E'' and $\Delta : E'' \longrightarrow \mathcal{A}$ a module homomorphism and Θ is the canonical map from E to E''. If E is $\Delta \circ \Theta$-amenable, then E'' is Connes-Δ-amenable.*

Proposition 2.11. *Assume that E is a Banach $\mathcal{A}$-module and E' a dual $\mathcal{A}$-module and $\Delta : E' \longrightarrow \mathcal{A}$ is a w^*-norm continuous module homomorphism and $\Theta : E \longrightarrow E'$ is a module homomorphism with w^*-dense range. If E' is Connes Δ-amenable then E is $\Delta \circ \Theta$ amenable.*

Proof. Let X is a Banach $\mathcal{A}$-module and $D : \mathcal{A} \longrightarrow X'$ a bounded linear map such that

$$D((\Delta \circ \Theta))(a \cdot x)) = a \cdot D((\Delta \circ \Theta)(x)) + (Da) \cdot (\Delta \circ \Theta)(x) \quad (a \in \mathcal{A},\ x \in E)\,.$$

X is Δ-normal and $D \circ \Delta : E' \longrightarrow X'$ is $w^* - w^*$ continuous. For $y \in E'$, there is a net $\{e_\alpha\}$ in E such that $\varphi(e_\alpha) \xrightarrow{w^*} y$. We have

$$\begin{aligned} D(\Delta(a \cdot y)) &= \lim_\alpha D(\Delta(a \cdot \Theta(e_\alpha))) \\ &= \lim_\alpha a \cdot D((\Delta \circ \Theta)(e_\alpha)) + \lim_\alpha D(a) \cdot (\Delta \circ \Theta)(e_\alpha) \\ &= a \cdot (D \circ \Delta)(y) + (Da) \cdot (\Delta(y)) . \end{aligned}$$

Since E' is Connes-Δ-amenable, there is $f \in X'$ such that $D(\Delta(y)) = (\Delta y) \cdot f - f \cdot (\Delta y)$. □

Corollary 2.12. *Let E is a Banach $\mathcal{A}$-module with second dual E'' and $\Delta : E'' \longrightarrow \mathcal{A}$ a w^*- norm continuous module homomorphism and Θ is the canonical map from E to E''. If E'' is Connes-Δ-amenable, then E is $\Delta \circ \Theta$-amenable.*

Example. For locally compact group G, let $A = L^1(G)$, $E = C_0(G)$ and $E' = M(G)$ for $f \in L^1(G)$ we define: $\Delta : M(G) \longrightarrow L^1(G)$, $\mu \longmapsto \mu * f$ and $\Theta : L^1(G) \hookrightarrow M(G)$. The maps Δ and Θ are bounded module homomorphism. $L^1(G)$ is w^*-dense in $M(G)$. Then $L^1(G)$ is $\Delta \circ \Theta$ amenable iff $M(G)$ is Connes-Δ-amenable.

References

[1] M. Amini, D. Ebrahimi Bagha, Module amenability for Banach modules, *Submited*.

[2] B.E. Johnson, Approximate diagonal and cohomology of certain annihilator Banach algebras, *Amer. J. Math.*, **94** (1972), 685-698.

[3] B.E. Johnson, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.*, **127** (1972).

[4] V. Runde, Amenability for dual Banach algebras, *Studia Math.*, To Appear.