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MODIFIED FAT POINTS OF PROJECTIVE VARIETIES AND POSTULATION

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Abstract: Let X be an integral n-dimensional projective variety, $P \in X_{reg}$, $m, e \in \mathbb{Z}$ such that $m > 0$ and $0 \leq e \leq \binom{n+m-1}{n-1}$. Here we introduce a class of connected zero-dimensional schemes sadwiched between the fat points mP and $(m+1)P$ and with length length$(mP) + e$. Then we study the postulation of a general union of such schemes.

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1. Modified Fat Points

For any scheme A and any $P \in A_{reg}$ and any integer $m > 0$ let mP (or $2m\{P, A\}$ if there is any danger of misunderstandings) denote the $(m-1)$-th infinitesimal neighborhood of P in A, i.e. the closed zero-dimensional subscheme of A with $(\mathcal{I}_P)^m$ as its ideal sheaf. We have length$(mP) = \binom{\dim_P(A)+m-1}{\dim_P(A)}$. We will say that mP is the double point of A with P as its support. Set $0P = 0\{P, A\} = \emptyset$. Fix $P \in A_{reg}$, an integer $m > 0$, an integer e such that $0 \leq e < \binom{n+m-1}{n-1}$, where $n := \dim_P(A)$; let μ be the maximal ideal of $\mathcal{O}_{A,P}$; fix a linear subspace $V \subseteq (\mu^m/\mu^{m+1})^*$ such that $\dim(V) = \binom{n+m-1}{n-1} - e$; notice that μ^m/μ^{m+1} is a $(\binom{n+m-1}{n-1})$-dimensional $\mathbb{K}$-vector space; there is a unique zero-dimensional

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scheme $Z[P,m,V] \subset A$ such that $m\{P,A\} \subseteq Z[P,m,V] \subset (m+1)\{P,a\}$ and the ideal sheaf of $Z[m,V]$ is generated by the ideal sheaf of $(m+1)\{P,A\}$ and by the germs of regular functions $f \in \mathcal{O}_{A,P}$ vanishing on $m\{P,A\}$ and whose Taylor expansion images in μ^m/μ^{m+1} are orthogonal to V. The proof of the main result in [5] easily gives the following result. It is easy to check that for fixed A, P, m, e the set of all such zero-dimensional schemes is irreducible and of dimension $e(\binom{n+m-1}{n-1} - e)$. If we keep fixed the integers m, e and vary the point P in A_{reg} we obtain an irreducible $ne(\binom{m+m-1}{n-1} - e)$-dimensional family of connected zero-dimensional subschemes of A. We will say that a scheme $Z[P,m,V]$ is a modified fat point of type (m,e) supported by P.

Proposition 1. *Let X be an integral n-dimensional projective variety, $E \subset X$ a zero-dimensional scheme, $\mu \in \mathbb{Z}$, $\mu > 0$ and $M, H \in \mathrm{Pic}(X)$ such that H is ample. Then there exists an integer $\alpha(X,H,M,A)$ (depending only from X, H,M,A) such that for every integer $x \geq \alpha(X,H,M,E)$ and integer $s > 0$ and all choices of integers m_i, e_i, $1 \leq i \leq s$, such that $0 < m_i \leq \mu$ and $0 \leq e_i < \binom{n+m-1}{n-1}$ the restriction map $H^0(X, \mathcal{I}_E \otimes M \otimes H^{\otimes x}) \to H^0(Z, (M \otimes H^{\otimes x})_{|Z})$ has maximal rank (i.e. either $h^1(X, \mathcal{I}_{E\cup Z} \otimes M \otimes H^{\otimes x}) = 0$ or $h^0(X, \mathcal{I}_{E\cup Z} \otimes M \otimes H^{\otimes x}) = 0$), where Z is a general union of s connected schemes $Z_i[P_i, m_i, V_i]$, $1 \leq i \leq s$, $\dim(V_i) = \binom{n+m-1}{n-1} - e_i$.*

Using induction on the integer s and [6], Theorem 1.4 (which is almost equivalent to the case $s = 1$), we immediately obtain the following result.

Theorem 1. *Let X be an integral n-dimensional projective variety. Fix integer $s \geq 1$, $m_i > 0$ and $0 \leq a_i \leq x_i$, $1 \leq i \leq s$. Let $Z_i \subset X$ be the union of x_i general fat points of multiplicities m_i of X and $W_i \subset X$ the general union of $x_i - a_i$ fat point with multiplicity m_i and a_i fat points of type $(m_i, 1)$. Set $Z := \cup_{i=1}^s Z_i$ and $W := \cup_{i=1}^s W_i$. Then $h^1(X, \mathcal{I}_W \otimes L) = h^1(X, \mathcal{I}_Z \otimes L) + \max\{0, \sum_{i=1}^s a_i - h^0(X, \mathcal{I}_Z \otimes L)\}$.*

As an immediate corollary of Theorem 1 and the classification of all exceptional cases for the postulation of general double points for any Veronese embedding of any projective space (see [1], [2], [3], [4], [6]) we obtain the following result.

Corollary 1. *Fix integers $n \geq 2$, $d \geq 3$, $x \geq 0$ and $y > 0$. Let $Z \subset \mathbf{P}^n$ be a general union of x double points and y points of type $(2,1)$. Then $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = \max\{0, (n+1)x + (n+2)y - \binom{n+d}{n}\}$ and $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = \max\{0, \binom{n+d}{n} - (n+1)x + (n+2)y\}$.*

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

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