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THE AUGMENTED LAGRANGIAN METHOD FOR THE PACKING OF UNEQUAL CIRCLES WITHIN A STRIP

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Abstract: This paper formulates the problem of packing a given set of different-sized circles into a strip as a nonlinear programming problem and establishes the first order optimality conditions. The augmented Lagrangian method is applied to solve this problem and the computational experiments show its effectiveness.

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1. Introduction

Given a set $I = \{1, \ldots, n\}$ of n circle items, each having radius r_i, and a strip of width W and infinite height, the two-dimensional circle-packing problem (2CP) consists of allocating all the items, without overlapping, to the strip by minimizing the overall height of the packing. Problem 2CP has many industrial applications, especially in cutting (wood and glass industries) and packing

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(transportation and warehousing). Although the circle packing is less complicated than the packing of rectangles, they are both NP-hard. Results of the circle-packing will form a firm foundation for the further investigation of the more complicated irregular polygon problems.

The literature on packing problems in general is extensive and too broad to be covered here. Several authors have studied the problem of packing circles of the same size into a rectangle including Dowsland [1], Fraser and George [2] and Isermann [5]. Isermann outlines a series of heuristics for packing homogenous circles in generality while Fraser and George look at this problem in the context of stowing rolls of paper products. Fraser and George discuss packing circles of the same size in a container of fixed dimensions whereas Dowsland addresses the problem of deciding the optimal size package in which to stow cylindrical products.

There are fewer publications discussing the problems of unequal circle packing. In a discrete manner, Hochbaum and Maass [4] considered the packing of objects with the same shape but different size. George et al [3] studied the packing of unequal circles within a square with an application to the transport of tubes. Lubachevsky and Graham [6] studied the problem of finding a smallest possible tray to contain a number of small circles having a unit radius.

To our knowledge, most results presented in the literature on 2CP concern heuristic and genetic algorithms. In this paper,we suggest a deterministic method for solving it. In Section 2 we formulate 2CP as a general optimization model and give the first-order optimality conditions. In Section 3, we transform 2CP into an unconstrained problem by the augmented Lagrangian method and present the promising results through computational experiments.

2. Mathematical Model

Let capital letter I represent the set of all circles, and i an elements of I. The radius and coordinates of the ith circle are r_i and (x_i, y_i) respectively. Consider the set of all different circle pairs, expressed as $H = \{(i,j) \mid i \in I, j \in I, i < j\}$, we can deduce $\mid H \mid= n(n-1)/2$, define $M =\mid H \mid$. With the symbols defined the problem of packing unequal circles within a strip is now modeled as

$$\begin{aligned}
&\text{minimize} \quad v\,, \\
&\text{subject to} \;\; (x_i - x_i)^2 + (y_i - y_i)^2 - (r_i + r_i)^2 \geq 0,\ (i,j) \in H, \qquad (1)\\
&r_i \leq x_i \leq W - r_i, \quad r_i \leq y_i \leq v - r_i, \quad i \in I.
\end{aligned}$$

In these expressions, r_i is a known real positive number, x_i, y_i, $i \in I$ and v are unknowns. Let $z = (x, y, v)$, we can convert this model into a general nonlinear programming problem by defining functions $f_0 : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}$ and $F : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^M \times \mathbf{R}^n$ as follows

$$f_0(z) = v, \quad F(z) = (F_1(z)^\top, F_2(z)^\top)^\top$$

with

$$\begin{aligned} F_1(z) =& ((x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 + r_2)^2, \cdots, (x_{n-1} - x_n)^2 \\ & + (y_{n-1} - y_n)^2 - (r_{n-1} + r_n)^2)^\top, \\ F_2(z) =& (v - y_1 - r_1, \cdots, v - y_n - r_n)^\top. \end{aligned}$$

Let $\Omega = \{z \in Z | F(z) \in D\}$, where $D = \mathbf{R}_+^M \times \mathbf{R}_+^n$ and

$$Z = \left\{ z \in \mathbf{R}^{2n+1} |\ r \leq x \leq We_{[n]} - r\,,\ y \geq r\,,\ v \in \mathbf{R}^n \right\}\,,$$

where $e_{[n]} = (1, \cdots, 1)^\top \in \mathbf{R}^n$. Then Model (1) can be equivalently formulated as a nonlinear programming problem:

Model NLP.

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & F(z) \in D, \\ & z \in Z. \end{array}$$

Let us introduce some notations for the purpose of deriving the first-order optimality conditions for Model NLP:

$$D_1 = \begin{pmatrix} d_{1,2}(x) & d_{1,3}(x) & \cdots & d_{1,n}(x) & 0 & \cdots & 0 & \cdots & 0 \\ -d_{1,2}(x) & 0 & \cdots & 0 & d_{2,3}(x) & \cdots & d_{2,n}(x) & \cdots & 0 \\ 0 & -d_{1,3}(x) & \cdots & 0 & -d_{2,3}(x) & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & d_{n-1,n}(x) \\ 0 & 0 & \cdots & -d_{1,n}(x) & 0 & \cdots & -d_{2,n}(x) & \cdots & -d_{n-1,n}(x) \end{pmatrix},$$

$$D_2 = \begin{pmatrix} c_{1,2}(x) & c_{1,3}(x) & \cdots & c_{1,n}(x) & 0 & \cdots & 0 & \cdots & 0 \\ -c_{1,2}(x) & 0 & \cdots & 0 & c_{2,3}(x) & \cdots & c_{2,n}(x) & \cdots & 0 \\ 0 & -c_{1,3}(x) & \cdots & 0 & -c_{2,3}(x) & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & c_{n-1,n}(x) \\ 0 & 0 & \cdots & -c_{1,n}(x) & 0 & \cdots & -c_{2,n}(x) & \cdots & -c_{n-1,n}(x) \end{pmatrix},$$

where $d_{i,j}(x) = 2(x_i - x_j)$, $c_{i,j}(y) = 2(y_i - y_j)$. Then $\nabla F_i(z)^\top$, $i = 1, 2$ can be expressed as

$$\nabla F_1(z)^\top = \begin{pmatrix} D_1 \\ D_2 \\ 0_{1\times M} \end{pmatrix}, \quad \nabla F_2(z)^\top = \begin{pmatrix} 0_{n\times n} \\ -I_{n\times n} \\ 1_{1\times n} \end{pmatrix}$$

and the transpose of the Jacobian of F is

$$\nabla F(z)^\top = (\nabla F_1(z)^\top, \nabla F_2(z)^\top).$$

Now we give the formulas for calculating the normal cones $N_Z(F(\bar{z}))$ and $N_D(F(\bar{z}))$ so that the formula $N_\Omega(\bar{z})$ can be derived for $\bar{z} \in \Omega$. Let

$$Z_x = \{x \in \mathbf{R}^n | r \le x \le W_{e[n]} - r\}, \quad Z_y = \{y \in \mathbf{R}^n | y \ge r\}$$

with $Z_x^i = [r_i, W - r_i]$ and $Z_y^i = [r_i, \infty)$. Then $N_Z(\bar{z}) = N_{Z_x}(\bar{x}) \times N_{Z_y}(\bar{y}) \times N_{\mathbf{R}}(\bar{v})$, where $N_{Z_x}(\bar{x}) = N_{Z_x^1}(\bar{x_1}) \times \cdots \times N_{Z_x^n}(\bar{x_n})$ with

$$N_{Z_x^i}(\bar{x_i}) = \begin{cases} \mathbf{R}_-, & \bar{x_i} = r_i; \\ \{0\}, & r_i < \bar{x_i} < W - r_i; \\ \mathbf{R}_+, & \bar{x_i} = W - r_i. \end{cases}$$

$N_{Z_y}(\bar{y}) = N_{Z_y^1}(\bar{y_1}) \times \cdots \times N_{Z_y^n}(\bar{y_n})$ with

$$N_{Z_y^i}(\bar{y_i}) = \begin{cases} \mathbf{R}_-, & \bar{y_i} = r_i; \\ \{0\}, & \bar{y_i} > r_i, \end{cases}$$

$N_{\mathbf{R}}(\bar{v}) = \{0\}$.

The normal cone of D at $F(\bar{z})$, $N_D(F(\bar{z}))$ is

$$N_D(F(\bar{z})) = N_{\mathbf{R}_+^M}(F_1(\bar{z})) \times N_{\mathbf{R}_+^n}(F_2(\bar{z})),$$

where

$$N_{\mathbf{R}_+^M}(F_1(\bar{z})) = N_{\mathbf{R}_+}(F_1^{1,2}(\bar{z})) \times \cdots \times N_{\mathbf{R}_+}(F_1^{n-1,n}(\bar{z})),$$

with

$$N_{\mathbf{R}_+}(F_1^{i,j}(\bar{z})) = \begin{cases} \mathbf{R}_-, & F_1^{i,j}(\bar{z}) = 0; \\ \{0\}, & F_1^{i,j}(\bar{z}) > 0, \end{cases}$$

$$N_{\mathbf{R}_+^n}(F_2(\bar{z})) = N_{\mathbf{R}_+}(F_2^1(\bar{z})) \times \cdots \times N_{\mathbf{R}_+}(F_2^n(\bar{z})),$$

with

$$N_{\mathbf{R}_+}(F_2^i(\bar{z})) = \begin{cases} \mathbf{R}_-, & F_2^i(\bar{z}) = 0; \\ \{0\}, & F_2^i(\bar{z}) > 0. \end{cases}$$

Lemma 1. (Normal Cone of Ω at $\bar{z}$) *Let $\bar{z} \in \Omega$, satisfy $r < \bar{x} < W - r$ and $\bar{y} > r$. Then the normal cone of Ω at $\bar{z}$ can be expressed as* $N_\Omega(\bar{z}) = \{\bigtriangledown F(\bar{z})^\top p + q | p \in N_D(F(\bar{z})), q \in N_Z(\bar{z})\}$.

Proof. According to Theorem 6.14 in Rockafellar and Wets [7], we only need to check the validity of the following constraint qualification

$$-\bigtriangledown F(\bar{z})^\top p \in N_Z(\bar{z}), p \in N_D(F(\bar{z})) \Rightarrow p = 0. \tag{2}$$

Let $p = (p^{1T}, p^{2T}) \in N_D(F(\bar{z}))$ then $-\bigtriangledown F(\bar{z})^\top p \in N_Z(\bar{z})$ can be written as $\bigtriangledown F_1(\bar{z})^\top(-p^1) + \bigtriangledown F_2(\bar{z})^\top(-p^2)$, or in the matrix form:

$$\begin{pmatrix} D_1 & 0_{n\times n} \\ D_2 & -I_{n\times n} \\ 0_{1\times M} & 1_{1\times n} \end{pmatrix} \begin{pmatrix} -p^1 \\ -p^2 \end{pmatrix} \in N_Z(\bar{z}),$$

that is

$$\begin{aligned} -D_1 p^1 &\in N_{Z_x}(\bar{x}), \\ -D_2 p^1 + p^2 &\in N_{Z_y}(\bar{y}), \\ \sum_{i=1}^{n}(-p_i^2) &\in N_{\mathbf{R}}(\bar{v}). \end{aligned}$$

In view of $p^2 \in N_{\mathbf{R}_+^{\mathbf{n}}}(F_2(\bar{z}))$, we know that $p^2 \leq 0_n$, while we obtain $\sum_{i=1}^{n}(-p_i^2) = 0$ from $N_{\mathbf{R}}(\bar{v}) = \{0\}$, therefore we must have $p^2 = 0_n$. Since $r < \bar{x} < W - r$ and $\bar{y} > r$, we have that $N_{Z_x}(\bar{x}) = \{0_n\}$ and $N_{Z_y}(\bar{y}) = \{0_n\}$. Thus we have $D_1 p^1 = 0$ and $D_2 p^1 = 0$. If $p^1 \neq 0$, we can assume $p^1_{i,j} \neq 0$, then we have $2(x_i - x_j)p^1_{i,j} = 0$ and $2(y_i - y_j)p^1_{i,j} = 0$, it implies $x_i = x_j$ and $y_i = y_j$, this condition fails for any two circles without overlapping. By now we have verified the correctness of (2) and the formula for calculating $N_\Omega(\bar{z})$ can be obtained by Theorem 6.14 of Rockafellar and Wets [7]. □

Theorem 1. (First-Order Optimality Conditions for Model NLP) *Let $\bar{z} \in \Omega$ be a local minimizer to Model NLP, satisfying $r < \bar{x} < W - r$ and $\bar{y} > r$. then there exists a vector $p \in N_D(F(\bar{z}))$ such that $-[\nabla f_0(\bar{z}) + \bigtriangledown F(\bar{z})^\top p] \in N_Z(\bar{z})$.*

Proof. It follows from Theorem 6.12 of Rockafellar and Wets [7] that $-\nabla f_0(\bar{z}) \in N_\Omega(\bar{z})$. Thus the conclusion is obvious from Lemma 1. □

3. A Numerical Algorithm

In model (1), all the constraints are inequalities, so we can simply prescribe model (1) as

$$\begin{array}{ll} \text{minimize} & f(z)\,, \\ \text{subject to} & c_i(z) \geq 0, \quad i = 1, \ldots, M + 4n. \end{array}$$

We suggest solving this constrained optimization problem by the augmented Lagrangian method based on solving a sequence of unconstrained optimization problem involving parameter $\sigma > 0$ and Lagrange multiplier λ:

$$\min P(z, \lambda, \sigma) = f(z) + \sum_{i=1}^{4n+M} \begin{cases} -\lambda_i c_i(z) + 0.5\sigma_i c_i^2(z), & c_i(z) < \lambda_i/\sigma_i; \\ -0.5\lambda_i^2/\sigma_i, & \text{otherwise.} \end{cases}$$

Algorithm ALM. *Step 1.* Given $z_1 \in \mathbf{R}^{2n+1}, \lambda^{(1)}, \sigma^{(1)} \in \mathbf{R}^{M+4n}$ and $\lambda^{(1)} > 0, \sigma^{(1)} > 0; \varepsilon \geq 0, k := 1.$

Step 2. Solve min $P(z, \lambda^{(k)}, \sigma^{(k)})$ for z_{k+1};
if $\| c^{(-)}(z_{k+1}) \|_\infty \leq \varepsilon$ (where $c_i^{(-)}(z_k) = \min\{0, c_i(z_k)\}$), then stop.

Step 3. For $i = 1, \cdots, M + 4n$, let

$$\sigma_i^{(k+1)} = \begin{cases} \sigma_i^{(k)}, & |c_i^{(-)}(z_{k+1})| \leq 1/4|c_i^{(-)}(z_k)|; \\ \max[10\sigma_i^{(k)}, k^2], & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$\lambda_i^{(k+1)} = \max\{\lambda_i^{(k)} - \sigma_i^{(k)} c_i(z_{k+1}), 0\}, \; i = 1, \cdots, M + 4n.$$

$k := k + 1$; go to Step 2.

This algorithm was coded in *C++* and run on a *Pentium III 1000 MHz* on a series of instances from the literature. The numerical experiments show that the algorithm can find better results to small and moderate-sized problems. We select three of them to show the results. Figures 1–3 are the geometrical patterns for circles corresponding to the solutions obtained by the algorithm involving up to 33, 40 and 43 items respectively, which show the very good behavior of the proposed augmented Lagrangian method.

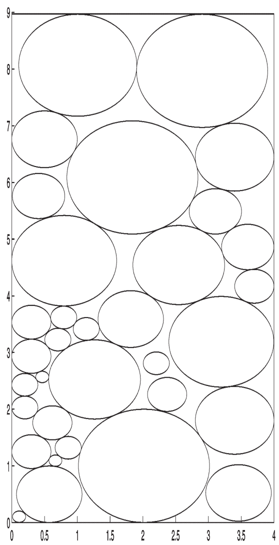

Figure 1

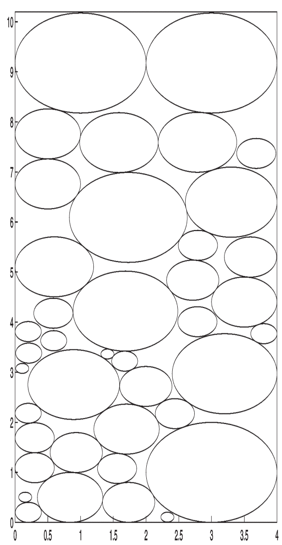

Figure 2

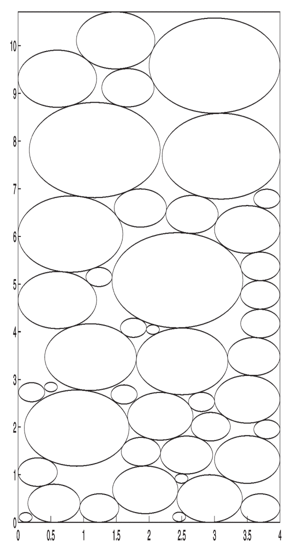

Figure 3

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