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THE SOLUTION CONCEPTS AND THE OPTIMALITY CONDITION FOR A KIND OF BILEVEL PROGRAMMING WITH MULTIFOLLOWERS

Lili Han[1 §], Rongsheng Li[2]
[1,2]Institute of Operation Research
Qufu Normal University
Rizhao, Shandong, 276826, P.R. CHINA
[1]e-mail: hllsd@eyou.edu.cn
[2]e-mail: lrs@qrnu.edu.cn

Abstract: In this paper, we introduce some new solution concepts for a bilevel programming problem with multiple followers based on the Pareto solution of multi-objective programming. We discuss the relation between the new solutions and the existing solutions and we give the optimality condition for a kind bilevel programming with multiple followers based on which, we can solve a class of bilevel programming with multiple followers via solving a corresponding bilevel programming with a single follower.

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1. Introduction

There is a certain kind of Stackelberg problem [1, 2, 4] with variety economical and strategy background in social life. In this kind of decision making problem, the Stackelberg decision rule is made between the upper and the lower, the

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§Correspondence author

multifollowers are coordinating reasonably among the followers. That is, they may cooperate with each other and take such decisions that favor some followers, so, on condition that upper level decision is fully respected, lower decision makers can take such decisions that most favor themselves in permitted range.

In this paper, we will concern with bilevel programming with multiple followers. In order to solve this kind of problems, we introduce new solution concepts based on the Pareto solution of multi-objective programming and discuss its properties. Then we give the optimality conditions, based on which, we can solve a class of bilevel programming with multiple followers via solving a corresponding bilevel programming with a single follower. It becomes to be simple and practical for solving this kind bilevel programming with multiple followers.

2. Solution Concepts and Properties

The bilevel programming with multiple followers problem can be formulated as (MBLP):

$$\text{(MBLP)} \qquad \begin{array}{ll} & \max\limits_{x\in X} F(x,y) \\ s.t. & \max\limits_{y_i\in Y_i} c_{i1}x + c_{i2}y_i \\ & G_i(x,y) \le 0 \qquad (i=1,2,\cdots,n), \end{array}$$

where $y = (y_1, y_2, \cdots, y_n); y_i \in Y_i \subseteq R^{m_i}; i = 1, 2, \cdots, n; x \in X \subseteq R^l$. Let $G(x) = \{(y_1, y_2, , \cdots, y_n)|G_i(x, y) \le 0, y \in Y_i \subseteq R^{m_i}, i = 1, 2, \cdots, n\}$, and $G(x) \neq \emptyset$. Here x and y_i are controlled by the upper and the i-th follower, respectively; $F(x, y)$ and $c_{i1}x + c_{i2}y_i$ are the objective functions of the upper and the i-th follower, respectively. The Stackelberg decision rule is made between the upper and the followers, the lower decision makers cooperate reasonably. First, the upper decision maker declare his decision x; second, the lower decision makers choose their decision variable y_i which favor their own objective functions. On condition that their own interests are assured, they may cooperate with each other, and take decision that favors some lower decision makers. Lower decisions also have effect on upper objective function, while upper decision makers may adjust their decision until their own objective functions were satisfied. At the same time, lower decision makers can adjust their own decision variable accordingly. Therefore, the decision making is a dynamic adjusting procedure.

By introducing Pareto solution [3, 5] concepts of multi-objective programming into the bilevel programming with multiple followers, we can have the new solution concepts as follows.

Definition 2.1. For any given $x \in X$, and for $y \in G(x)$ if does not exist $y' \in G(x)$ satisfying the following conditions:

$$c_{i1}x + c_{i2}y_i' \geq c_{i1}x + c_{i2}y_i, \quad i = 1, 2, \cdots, n$$

and there exist at least $i_0 \in \{1, 2, \cdots, n\}$ such that

$$c_{i_0 1}x + c_{i_0 2}y_{i_0} > c_{i_0 1}x + c_{i_0 2}y_{i_0},$$

then y is called Pareto solution of the followers with respect to x.

Definition 2.2. For any given $x \in X$, if $y \in G(x)$ and for $\forall \ y' \in G(x)$ satisfying the following conditions:

$$c_{i1}x + c_{i2}y_i \geq c_{i1}x + c_{i2}y_i', \quad i = 1, 2, \cdots, n \tag{2.1}$$

and there exist at least $i_0 \in \{1, 2, \cdots, n\}$ satisfying

$$c_{i_0 1}x + c_{i_0 2}y_{i_0} > c_{i_0 1}x + c_{i_0 2}y_{i_0}', \tag{2.2}$$

then y is called an efficient Pareto solution of the followers with respect to x.

For $x \in X$, we denote by $P(x)$ and $EP(x)$ the Pareto solution and the efficient Pareto solution set.

Definition 2.3. For $x \in X$, if $y \in P(x)$, then (x, y) is called a Pareto feasible solution of (MBLP). Denote the feasible Pareto solution set of (MBLP) by F_{MBLP}.

Definition 2.4. For $(x, y) \in F_{MBLP}$, if $F(x, y) \geq F(x', y')$ for $\forall (x', y') \in F_{MBLP}$, then (x, y) is called a satisfactory solution of (MBLP).

In order to discuss the programming (MBLP), we introduce a class of bilevel programming with a single follower (SBLP) as follows:

$$\text{(SBLP)} \qquad \begin{array}{ll} & \max\limits_{x \in X} F(x, y) \\ s.t. & \max\limits_{y \in Y} \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i \\ & G_i(x, y) \leq 0 \qquad (i = 1, 2, \cdots, n), \end{array}$$

where $y = (y_1, y_2, \cdots, y_n;\ Y = Y_1 \times Y_2 \times \cdots \times Y_n;\ y_i \in Y_i \subseteq R^{m_i};\ i = 1, 2, \cdots, n;\ x \in X \subseteq R^l$.

Denote the feasible Pareto solution set of programming (SBLP) by F_{SBLP}.

Definition 2.5. For any $x \in X$, if $y \in G(x)$ and $\sum_{i=1}^{n} c_{i1}x + c_{i2}y_i \geq \sum_{i=1}^{n} c_{i1}x + c_{i2}y'_i$ for $\forall y' \in G(x)$, then (x, y) is called a feasible solution of (SBLP). Denote the feasible solution set by F_{MBLP}.

Definition 2.6. If $(x, y) \in F_{SBLP}$ such that $F(x, y) \geq F(x', y')$ for $\forall(x', y') \in F_{SBLP}$, then (x, y) is called a satisfactory solution of (SBLP).

We denote the Nash equilibrium reaction set [6] of the followers with respect to the superior strategy by $NE(x)$.

3. Main Results

We give the properties of the Pareto solution of (MBLP) as follows.

Theorem 3.1. *For each $x \in X$, $EP(x) \subseteq P(x)$.*

Proof. We will discuss Theorem 3.1 in two cases:

Obviously, we can assume that $EP(x \neq \Phi$, let $y \in EP(x)$. By Definition 2.2, we can see that $y \in G(x)$ and for $\forall y' \in G(x)$, the following conditions (2.1) and (2.2) hold.

If $y \neq P(x)$, then by Definition 2.1, there exists $y^0 \in G(x)$ satisfying $c_{i1}x + c_{i2}y_i^0 \geq c_{i1}x + c_{i2}y_i$ and there exists $i_0 \in \{1, 2, \cdots, n\}$ such the inequality holds strictly, which contradicts with (2.1)(2.2), hence $y \in P(x)$. □

Theorem 3.2. *For any given $x \in X$, if $y \in P(x)$, then $y \in NE(x)$.*

Proof. For any given $x \in X$, let $y = (y_1, y_2, \cdots, y_{i_0-1}, y_{i_0}, y_{i_0+1}, \cdots, y_n)$ and $y \in P(x)$. If y is not in the Nash equilibrium reaction set [6] of the followers with respect to the superior strategy x, then by the definition of Nash equilibrium reaction set, we know that there exists $y' = (y_1, y_2, \cdots, y_{i_0-1}, y'_{i_0}, y_{i_0+1}, \cdots, y_n) \in P(x)$ satisfying the following condition:

$$c_{i_0 1}x + c_{i_0 2}y'_{i_0} > c_{i_0 1}x + c_{i_0 2}y_{i_0} ,$$

$$c_{i1}x + c_{i2}y'_i = c_{i1}x + c_{i2}y_i \quad (i = 1, 2, \cdots, i_0 - 1, i_0 + 1, \cdots, n) .$$

This contradicts that y is Pareto solution of the lower with respect to x in the bilevel programming(MBLP). Thus y is the Nash equilibrium reaction of the followers with respect to the superior strategy x. So $y \in NE(x)$. □

Corollary 3.1. *For any given $x \in X$, $y \in EP(x)$, then $y \in NE(x)$.*

Theorem 3.3. *For any given $x \in X$, the Pareto solution set of the follower with respect to x is not empty, namely, $P(x) \neq \Phi$.*

Proof. Since $G(x) \neq \Phi$, if $P(x) = \Phi$, for any fixed $y \in G(x)$, we try to prove that there exists $y^* \in G(x)$ such that

$$c_{i1}x + c_{i2}y_i^* \geq c_{i1}x + c_{i2}y_i \quad (i = 1, 2, \cdots, n) \tag{3.1}$$

and there exists at least one $i_0 \in (1, 2, \cdots, n)$ such that

$$c_{i_01}x + c_{i_02}y_{i_0}^* > c_{i_01}x + c_{i_02}y_{i_0}\,. \tag{3.2}$$

At the same time there does not exist $y' \in G(x)$ such that

$$c_{i1}x + c_{i2}y_i' \geq c_{i1}x + c_{i2}y_i^* \quad (i = 1, 2, \cdots, n) \tag{3.3}$$

and there exists $i_0 \in (1, 2, \cdots, n)$ satisfying

$$c_{i_01}x + c_{i_02}y_{i_0}' > c_{i_01}x + c_{i_02}y_{i_0}^*\,. \tag{3.4}$$

Now, we assume that there does not exist y^* satisfying (3.1)-(3.4), and since $G(x) \neq \Phi$, if there does not exist y^* satisfying condition (3.1) and (3.2). By Definition 2.1, we know $y \in P(x)$. This contradicts with $P(x) = \Phi$. So we can suppose that there exists $y^* \in G(x)$ satisfying (3.1) and (3.2), but not satisfying (3.3) and (3.4), it implies that there exists $y' \in G(x)$ satisfying

$$c_{i1}x + c_{i2}y_i' \geq c_{i1}x + c_{i2}y_i \quad (i = 1, 2, \cdots, n)$$

and there exists $i_0 \in (1, 2, \cdots, n)$ such that

$$c_{i_01}x + c_{i_02}y_{i_0}' > c_{i_01}x + c_{i_02}y_{i_0}\,.$$

Let $y^* = y'$, we get y^* which satisfies (3.1)-(3.4). Noting Definition 2.1, y^* is an efficient Pareto solution of the lower with respect to the superior strategy x of the bilevel programming (MBLP), that is, $y^* \in P(x)$. It contradicts with $P(x) = \Phi$. To sum up, the conclusion of the Theorem 3.3 holds. □

By Theorem 3.2 and Theorem 3.3, we can get the following conclusion.

Corollary 3.2. *For any given $x \in X$, the Nash equilibrium reaction set of the follower with respect to x is not empty, namely, $NE(x) \neq \Phi$.*

Lemma 3.1. *If for any given $x \in X$, the efficient Pareto solution set of the followers with respect to x is nonempty, $P(x) = EP(x)$.*

Proof. First, we show that $EP(x) \subseteq P(x)$. For any given $x \in X$, if $y \in EP(x)$, by Theorem 3.1, $y \in P(x)$.

Next, we show $P(x) \subseteq EP(x)$. For any fixed $x \in X$, if $y \in P(x)$ and $y \notin EP(x)$. Noting $y \in P(x)$, we can know that there does not exist $y' \in G(x)$ which satisfies the following two conditions:

$$c_{i1}x + c_{i2}y'_i \geq c_{i1}x + c_{i2}y_i \quad (i = 1, 2, \cdots, n)\,, \tag{3.5}$$

there exists at least one $i_0 \in \{1, 2, \cdots, n\}$ such that

$$c_{i_0 1}x + c_{i_0 2}y'_i > c_{i_0 1}x + c_{i_0 2}y_{i_0}\,. \tag{3.6}$$

Since $EP(x) \neq \Phi$ and $y \notin EP(x)$, so there exists at least one $y^* \in EP(x)$ and satisfying the following conditions:

$$c_{i1}x + c_{i2}y^*_i \geq c_{i1}x + c_{i2}y_i \quad (i = 1, 2, \cdots, n)$$

and there exists $i_0 \in \{1, 2, \cdots, n\}$ such that

$$c_{i_0 1}x + c_{i_0 2}y^*_i > c_{i_0 1}x + c_{i_0 2}y_{i_0}\,.$$

It contradicts with that there does not exists $y' \in G(x)$ satisfying (3.5) and (3.6). Thus $y \in EP(x)$.

Therefore, the conclusion of Lemma 3.1 holds. □

On the relation between the efficient Pareto solution set and the Nash equilibrium reaction set of the follower with respect to x, we can obtain the following conclusion.

Theorem 3.4. *For any given $x \in X$, if the Nash equilibrium reaction set of the follower with respect to x is singleton, then $y \in EP(x)$ if and only if $y \in NE(x)$.*

Proof. ($\Rightarrow$) For any fixed $x \in X$, if $y \in P(x)$, then by Theorem 3.2, $y \in EP(x)$.

($\Leftarrow$) If $y \in NE(x)$ then y is the Nash equilibrium reaction of the follower with respect to x, and by Theorem 3.3, we can get $P(x) \neq \Phi$. If $y \notin P(x)$, then there exists $y' \in P(x)$ and $y' \neq y$. From Theorem 3.2, we can obtain $y' \in NE(x)$, thus $y, y' \in NE(x)$. It contradicts that $NE(x)$ is a singleton, so $y \in P(x)$. □

We can get the conclusion of the efficient Pareto solution set and the Nash equilibrium reaction set of the followers.

Theorem 3.5. *For any given $x \in X$, if $EP(x) \neq \Phi$, then $y \in P(x)$ if and only if y is the optimal solution of programming (LP) with respect to x.*

$$(LP) \qquad s.t. \qquad \begin{array}{l} \max\limits_{y\in Y} \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i \\ G_i(x,y) \leq 0 \quad (i = 1, 2, \cdots, n), \end{array}$$

where $y = (y_1, y_2, \cdots, y_n) : Y = Y_1 \times Y_2 \times \cdots \times Y_n; y_i \in Y_i \subseteq R^{m_i}; i = 1, 2, \cdots, n; x \in X \subseteq R^l$.

Proof. First, for any given $x \in X$, if $EP(x) \neq \Phi$, then by Lemma 3.1, we can only prove that $y \in EP(x)$ if and only if y is an optimality solution of respect with x of programming (LP).

($\Rightarrow$) For any given $x \in X$, y is an effective Pareto solution in lower level, namely, $y \in EP(x)$ and $y \in G(x)$. If y is not an optimum solution of single level programming (LP) corresponding to x, then we would suppose y^* is the optimality solution of respect with x of programming (LP). It holds $y^* \in G(x)$. Noting $y \in EP(x)$, by Definition 2.2, we can obtain that $c_{i1}x + c_{i2}y_i \geq c_{i1}x + c_{i2}y_i^*$ and there exists $i_0 \in \{1, 2, \cdots, n\}$ such that $c_{i_01}x + c_{i_02}y_{i_0} > c_{i_01}x + c_{i_02}y_{i_0}^*$. So $\sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i > \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i^*$, it contradicts that y^* is the optimality solution of respect with x of programming (LP) respective x. So y is an optimality solution of programming (LP).

($\Leftarrow$) For any fixed $x \in X, y$ is the optimality solution with respect to x, of programming (LP), then $y \in G(x)$ and for $\forall y' \in G(x)$ such that

$$\sum_{i=1}^{n} c_{i1}x + c_{i2}y_i > \sum_{i=1}^{n} c_{i1}x + c_{i2}y_i' . \tag{3.7}$$

If $y \notin EP(x)$, since $EP(x) \neq \Phi$, so there exists $y^* \in EP(x)$ and $y^* \neq y$, by Definition 2.2, $y^* \in G(x)$ and for $\forall y'' \in G(x)$ such that $c_{i1}x + c_{i2}y_i^* \geq c_{i1}x + c_{i2}y'', (i = 1, 2, \cdots, n)$ and there exists at least one $i_0 \in \{1, 2, \cdots, n\}$ satisfying $c_{i_01}x + c_{i_02}y_{i_0}^* > c_{i_01}x + c_{i_2}y''$. Since $y \in G(x)$, thus $\sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i^* > \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i$. It contradicts with (3.7), so y is an effective Pareto solution in lower level for any fixed x, thus $y \in EP(x)$. $\square$

Corollary 3.3. *For any given $x \in X$, if y is an optimal solution of programming (LP) with respect to x then $y \in P(x)$.*

Proof. For any $x \in X, y$ is an optimal solution of programming (LP) with respect to x, then $y \in G(x)$ and for $\forall y' \in G(x)$ there exists

$$\sum_{i=1}^{n} c_{i1}x + c_{i2}y_i > \sum_{i=1}^{n} c_{i1}x + c_{i2}y'_i. \tag{3.8}$$

If $y \notin P(x)$, it implies $y^* \in G(x)$ such that

$$c_{i1}x + c_{i2}y_i^* \geq c_{i1}x + c_{i2}y_i \quad (i = 1, 2, \cdots, n)$$

and there exists $i_0 \in \{1, 2, \cdots, n\}$ satisfying $c_{i_01}x + c_{i_02}y_{i_0}^* > c_{i_01}x + c_{i_02}y_{i_0}$, thus $\sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i^* > \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i$. It contradicts with (3.8), so $y \in P(x)$.□

From above, we can get the conclusion of the relation of the feasible set of (MBLP) and (SBLP).

Theorem 3.6. *For any given $x \in X$, if $EP(x) \neq \Phi$, then F_{MBLP} is equal to F_{SBLP}.*

Proof. First, we show that $F_{MBLP} \subseteq F_{SBLP}$.

For $(x, y) \in F_{MBLP}$, i.e., for any fixed $x \in X$ and $y \in P(x)$, since $EP(x) \neq \Phi$. It follows from Lemma 3.1 that we can get $y \in EP(x)$. By Theorem 3.3, y is an optimal solution of programming (LP) with respect to x. which implies that $\sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i \geq \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y'_i$ for $\forall y' \in G(x)$. By Definition 2.5, (x, y) is a feasible solution of programming (SBLP), and then $(x, y) \in F_{SBLP}$.

Second, we will prove that $F_{SBLP} \subseteq F_{MBLP}$.

For $\forall (x, y) \in G_{SBLP}$, by Definition 2.5, we get that $x \in X$ and $y \in G(x)$, and for $\forall (x, y) \in G(x)$, it holds that $\sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y_i \geq \sum\limits_{i=1}^{n} c_{i1}x + c_{i2}y'_i$, consequently y is an optimal solution of programming (LP) with respect to x. Noting $EP(x) \neq \Phi$, we can know $y \in EP(x)$ from Theorem 3.5. By Definition 2.3, we can show that (x, y) is a feasible solution of bilevel (MBLP), so $(x, y) \in F_{MBLP}$.
□

According to bilevel programming (MBLP) and (SBLP) and Theorem 3.6, we can get the optimality condition which can solve the bilevel programming with multiple followers (MBLP) via solving the bilevel programming with single follower (SBLP) as follows.

Theorem 3.7. *For any given $x \in X$, if $EP(x) \neq \Phi$,then (x^*, y^*) is the satisfactory solution of (MBLP) if and only if is (x^*, y^*) an optimal solution of (SBLP).*

The conditions of Theorem 3.5 is satisfied by the bilevel programming with multiple followers, solving the bilevel programming with multiple followers via solving the bilevel programming with single follower, so its solving becomes easy and simple. The solution method is very practical.

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