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ADJACENT-VERTEX-DISTINGUISHING TOTAL CHROMATIC NUMBERS ON MONO-CYCLE GRAPHS AND THE SQUARE OF CYCLES

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Abstract: Let G be a simple graph. Let f be a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$. Let $C_f(v) = \{f(v)\} \cup \{f(vw)|w \in V(G), vw \in E(G)\}$ for every $v \in V(G)$. If f is k-proper-total-coloring, and for $u, v \in V(G), uv \in E(G)$, we have that $C_f(u) \neq C_f(v)$, then f is called the k-adjacent-vertex-distinguishing total coloring ($k - AVDTC$ for short). Let $\chi_{at}(G) = \min\{k|G$ has a k-adjacent-vertex-distinguishing total coloring$\}$. Then $\chi_{at}(G)$ is called the adjacent-vertex-distinguishing total chromatic number. The adjacent-vertex-distinguishing total chromatic number on mono-cycle graphs and the square of cycles are obtained in this paper.

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1. Introduction

The graphs considered in this paper are connected, limited, undirected, simple graphs. A k-proper-total-coloring of a graph G is a mapping f from $V(G) \cup E(G)$

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to $\{1, 2, \cdots, k\}$ such that:

1) $\forall u, v \in V(G)$, if $uv \in E(G)$, then $f(u) \neq f(v)$;

2) $\forall e_1, e_2 \in E(G), e_1 \neq e_2$, if e_1, e_2 have common end vertex, then $f(e_1) \neq f(e_2)$;

3) $\forall u \in V(G), e \in E(G)$, if u is the end vertex of e, then $f(u) \neq f(e)$.

Let f be a k-proper-total-coloring of G. Let $C_f(u) = \{f(u)\} \cup \{f(uw)|w \in V(G), uw \in E(G)\}$ and $\overline{C}_f(u) = \{1, 2, \cdots, k\} - C_f(u)$ for every $u \in V(G)$. If $\forall u, v \in V(G), uv \in E(G)$, we have $C_f(u) \neq C_f(v)$,i.e., $\overline{C}_f(u) \neq \overline{C}_f(v)$, then f is called a k-adjacent-vertex-distinguishing total coloring ($k - AVDTC$ for short). The number $\min\{k|G$ has a k-adjacent-vertex-distinguishing total-coloring $\}$ is called the adjacent-vertex-distinguishing total chromatic number and is denoted by $\chi_{at}(G)$.

The theory of vertex-distinguishing proper edge-coloring has been investigated in several papers [1, 2, 3, 4]. Adjacent strong edge coloring (i.e., adjacent-vertex-distinguishing proper edge-coloring) is considered in [7] by Zhongfu Zhang et al. The concept about the adjacent-vertex-distinguishing total coloring is proposed by Zhongfu Zhang and Xiang-en Chen et al in [6]. And the adjacent-vertex-distinguishing total coloring on cycle, complete graph, complete bipartite graph, fan, wheel and tree are discussed in [6]. According to these results, for adjacent-vertex-distinguishing total chromatic number, a conjecture and an open problem are given in [6].

Conjection 1.1. (see [6]) *For every connected graph G with order at least 2,we have* $\chi_{at}(G) \leq \Delta(G) + 3$.

Open Problem 1.2. (see [6]) When do we have $\chi_{at}(H) \leq \chi_{at}(G)$ if H is a subgraph of G?

In this paper, the adjacent-vertex-distinguishing total coloring on mono-cycle graphs and the square of cycles are studied and the corresponding chromatic number is obtained in Section 2 and Section 3, respectively. Theorem 2.2 and Theorem 3.1 in this paper will indicate that Conjecture 1.1 is valid for mono-cycle graphs and the square of cycles.

The following lemma is obvious.

Lemma 1.3. (see [6]) *If G does not have two distinct vertices of maximum degree which are adjacent, then* $\chi_{at}(G) \geq \Delta(G)+1$; *If G has two distinct vertices of maximum degree which are adjacent, then* $\chi_{at}(G) \geq \Delta(G) + 2$.

For the graph-theoretic terminology the reader is referred to [5].

2. The Adjacent-Vertex-Distinguishing Total Chromatic Number on Mono-Cycle Graphs

Let T be a tree with order $p(\geq 3)$. Suppose $u, v \in V(T)$ and u is not adjacent to v in T. The new graph G obtained by adding a new edge to T s.t. the new edge connects u and v is called the mono-cycle graph. Obviously for mono-cycle graph G, we have that $|V(G)| = |E(G)|$, G is connected and G has a unique cycle. In this section, if we deal with $k - AVDTC$, then the corresponding set composed of all k colors is $C = \{1, 2, \cdots, k\}$.

In order to discuss the adjacent-vertex-distinguishing total chromatic number on mono-cycle graphs, we give the following lemma firstly.

Lemma 2.1. *If a mono-cycle graph G has vertices of degree one, the adjacent vertices of all vertices of degree one are in the unique cycle C, the degrees of all vertices in C are equal, then $\chi_{at}(G) = \Delta(G) + 2$.*

Proof. Suppose $C = v_1v_2v_3 \cdots v_{n-1}v_nv_1$, $t = \Delta(G)-2$. $V(G) = \{v_1, v_2, \cdots, v_n\} \cup \cup_{i=1}^{n}\{v_{i1}, v_{i2}, \cdots, v_{it}\}$. $E(G) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1\} \cup \cup_{i=1}^{n}\{v_iv_{i1}, v_iv_{i2}, \cdots, v_iv_{it}\}$.

From Lemma 1.3, we have that $\chi_{at}(G) \geq \Delta(G) + 2$. In the following we only prove that G has a $(\Delta(G) + 2) - AVDTC$.

There are two cases to be considered.

Case 1. $n = 3$. Construct a mapping from $V(G) \cup E(G)$ to $\{1, 2, 3, \cdots, \Delta(G) + 2\}$ as follows.

$$f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_1v_2) = 3, f(v_2v_3) = 4, f(v_3v_1) = 5;$$
$$f(v_1v_{11}) = 4, f(v_2v_{21}) = 1, f(v_3v_{31}) = 2,$$
$$f(v_{1j}) = 2, f(v_{2j}) = 5, f(v_{3j}) = 1, j = 1, 2, \cdots, t;$$
$$f(v_iv_{i2}) = 6, f(v_iv_{i3}) = 7, \cdots, f(v_iv_{it}) = t + 4 = \Delta(G) + 2, i = 1, 2, 3.$$

Obviously f is a $(\Delta(G) + 2) - AVDTC$ of G. So $\chi_{at}(G) = \Delta(G) + 2$.

Case 2. $n \geq 4$. As $\chi_{at}(C_n) = 4$ (See [6]), therefore C_n has a $4 - AVDTCf$ with 4 colors $1, 2, 3, 4$. Based on this, define $f(v_iv_{ij}) = j + 4, j = 1, 2, \cdots, t, i = 1, 2, \cdots, n$. Let

$$C(v_i) = \{f(v_i), f(v_iv_{i+1}), f(v_iv_{i-1}), f(v_iv_{i1}), f(v_iv_{i2}), \cdots, f(v_iv_{it})\}, \quad i = 1, 2, \cdots, n.$$

Note that $v_0 = v_n, v_1 = v_{n+1}$. Assign the unique color which is not in $C(v_i)$ to all vertices $v_{i1}, v_{i2}, \cdots, v_{it}$, $i = 1, 2, \cdots, n$. By this way, we can obtain a $(\Delta(G) + 2) - AVDTC$ of G. So $\chi_{at}(G) = \Delta(G) + 2$.

The proof is completed. □

Theorem 2.2. *Let G be a mono-cycle graph, $p(G) = |V(G)| \geq 4$. If G does not have two distinct vertices of maximum degree of which are adjacent, then $\chi_{at}(G) = \Delta(G) + 1$. If G has two distinct vertices of maximum degree which are adjacent, then $\chi_{at}(G) = \Delta(G) + 2$.*

Proof. We prove this theorem by induction on order $p(G) = |V(G)|$.

If $p(G) = 4$, then $G = C_4$ or $G = G_1$, where $V(G_1) = \{v_1, v_2, v_3, v_4\}$, $E(G_1) = \{v_1v_2, v_2v_3, v_3v_1, v_1v_4\}$. From Lemma 1.3, we know that $\chi_{at}(G) \geq 4$. When $G = C_4 = v_1v_2v_3v_4v_1$. Obviously $\chi_{at}(G) = 4$ (see [6]). When $G = G_1$. Let $g(v_i) = i, i = 1, 2, 3, 4, g(v_2v_3) = 1, g(v_1v_2) = 3, g(v_3v_1) = 4, g(v_1v_4) = 2$. Then g is a $4 - AVDTC$ of G_1. So $\chi_{at}(G_1) = 4$.

Assume that the theorem is valid for mono-cycle graphs with order less than p. Now we consider the mono-cycle graph G with order $|V(G)| = p(\geq 5)$. The unique cycle in G is denoted by C. If G has no vertices of degree one, then $G = C \neq C_3$. So $\chi_{at}(G) = \Delta(G) + 2$ (see [6]).

If G has at least one vertex of degree one, then we have $\chi_{at}(G) \geq \Delta(G) + 1$ if G does not have two vertices of maximum degree which are adjacent, or $\chi_{at}(G) \geq \Delta(G)+2$ if G has two vertices of maximum degree which are adjacent. We need only to prove that G has $(\Delta(G)+1)-AVDTC$ if arbitrary two vertices of maximum degree of G are not adjacent or $(\Delta(G) + 2) - AVDTC$ if G has two vertices of maximum degree which are adjacent in the following. To this end, there are two cases to be considered.

Case 1. There exists a vertex v of degree one in G such that the adjacent vertex w of $v's$ is not in C. Let $G' = G - v$.

Case 1.1. $d_G(w) = \Delta(G)$.

Case 1.1.1. G does not have two vertices of maximum degree which are adjacent.

In this time, G' has a $(\Delta(G) + 1) - AVDTCf$ by the induction hypothesis (note that a $\Delta(G) - AVDTC$ is also a $(\Delta(G) + 1) - AVDTC$). Suppose $\overline{C}_f(w) = \{l_1\}, l_2 \in C - \{l_1, f(w)\}$. Define $f(vw) = l_1, f(v) = l_2$. Then the extended f is a $(\Delta(G) + 1) - AVDTC$ of G.

Case 1.1.2. There exist two vertices of maximum degree in G which are adjacent.

In this time, G' has a $(\Delta(G) + 2) - AVDTCf$ by the induction hypothesis (note that a $(\Delta(G) + 1) - AVDTC$ is also a $(\Delta(G) + 2) - AVDTC$). Suppose $N_G(w) = \{v, u_1, u_2, \cdots, u_r\}$,$d_G(w) = r + 1 = \Delta(G)$. Let

$$A = \{u_i | d_G(u_i) = \Delta(G) \text{ and } C_f(w) \subseteq C_f(u_i), i = 1, 2, \cdots, r\}.$$

Let $|A| = s$. Without loss of generality, we assume that $A = \{u_1, u_2, \cdots, u_s\}$.

If $s = 0, 1$, then we may obtain a $(\Delta(G)+2) - AVDTC$ of G easily. Assume $s \geq 2$ and $C_f(u_i) = C_f(w) \cup \{l_i\}, i = 1, 2, \cdots, s$.

Obviously $f(u_i w) \neq l_j$ for all $i = 1, 2, \cdots, r, j = 1, 2, \cdots, s$. Since $w \in V(G) - V(C)$, we know that there are exactly r components in $G' - w$. The component which contains u_i is denoted by G_i, $i = 1, 2, \cdots, r$. For the $(\Delta(G)+2) - AVDTCf$ of G', exchange the color l_i and color l_1 in G_i, $i = 1, 2, \cdots, s$. In this way, we can obtain a new $(\Delta(G)+2) - AVDTCg$ of G'. And for this new $(\Delta(G)+2) - AVDTCg$, we have that $C_g(u_1) = C_g(u_2) = \cdots = C_g(u_s) = C_g(w) \cup \{l_1\} = C_f(w) \cup \{l_1\}$. As $|C_g(u_i)| = d_G(u_i)+1 = d_G(w)+1 = \Delta(G)+1$, therefore there exists $c \in \overline{C}_g(u_i), i = 1, 2, \cdots, s$. Define $f(vw) = c, f(v) = l_1$. Then the extended f is a $(\Delta(G)+2) - AVDTC$ of G.

Case 1.2. $d_G(w) < \Delta(G)$.

Case 1.2.1. G does not have two vertices of maximum degree which are adjacent.

In this moment, $\Delta(G') = \Delta(G)$ and G' does not have two vertices of maximum degree which are adjacent. From the induction hypothesis, we know that G' has a $(\Delta(G)+1) - AVDTCf$. Let $N_G(w) = \{v, u_1, u_2, \cdots, u_r\}, A = \{u_i | C_f(w) \subseteq C_f(u_i), |C_f(u_i)| = r+2, i = 1, 2, \cdots, r\}$. Without loss of generality, we assume that $A = \{u_1, u_2, \cdots, u_s\}, 0 \leq s \leq r$. $d_G(w) = r+1 < \Delta(G)$.

Case 1.2.1.1. $s = 0$. Choose two distinct colors $l_1, l_2 \in \{1, 2, \cdots, \Delta(G)+2\} - C_f(w)$. Let $f(v) = l_1, f(vw) = l_2$, then the extended f is a $(\Delta(G)+2) - AVDTC$ of G.

Case 1.2.1.2. $s = 1$. In fact, there is only one color in $C_f(u_1) - C_f(w)$. Suppose $l_1 \in C_f(u_1) - C_f(w), l_2 \in \{1, 2, \cdots, \Delta(G)+1\} - C_f(u_1)$. Let $f(v) = l_1, f(vw) = l_2$, then the extended f is a $(\Delta(G)+1) - AVDTC$ of G.

Case 1.2.1.3. $s \geq 2$. Suppose $l_i \in C_f(u_i) - C_f(w), i = 1, 2, \cdots, s$. Obviously $f(u_i w) \neq l_j$ for all $i = 1, 2, \cdots, r,\ j = 1, 2, \cdots, s$. Since $w \in V(G) - V(C)$, therefore there are exactly r components in $G' - w$. The component which contain u_i is denoted by $G_i, i = 1, 2, \cdots, r$. For the $(\Delta(G)+1) - AVDTCf$ of G', exchange the color l_i and color l_1 in $G_i, i = 2, 3, \cdots, s$. In this way, we can obtain a new $(\Delta(G)+1) - AVDTCg$. And for the new $(\Delta(G)+1) - AVDTCg$, we have $C_g(u_1) = C_g(u_2) = \cdots = C_g(u_s) = C_g(w) \cup \{l_1\} = C_f(w) \cup \{l_1\}$. Since $|C_g(u_i)| = r+2 \leq \Delta(G), i = 1, 2, \cdots, s$, then there exists a color $t \in \{1, 2, \cdots, \Delta(G)+1\} - C_g(u_1) = \{1, 2, \cdots, \Delta(G)+1\} - C_f(u_1)$. Let $g(vw) = t, g(v) = l_1$, then g is a $(\Delta(G)+1) - AVDTC$ of G.

Case 1.2.2. There exist two vertices of maximum degree in G which are adjacent.

In this moment, $\Delta(G') = \Delta(G)$ and there exist two vertices of maximum degree in G' which are adjacent. From the induction hypothesis, we know that G' has a $(\Delta(G)+2) - AVDTCf$. Use the method completely similar to Case 1.2.1, we can obtain a $(\Delta(G)+2) - AVDTC$ of G.

Case 2. The adjacent vertices of all vertices of degree one are all in $V(C)$.

Case 2.1. There exist two vertices u_1, u_2 in $V(C)$ s.t. $d_G(u_1) \geq 3, d_G(u_2) \geq 3$ and $d_G(u_1) \neq d_G(u_2)$.

In this time, there exists $w, w_1, w_2 \in V(C)$, s.t. $ww_1, ww_2 \in E(G) \cap E(C), d_G(w_1) \neq d_G(w) \geq 3$. Suppose $v \in V(G)$, s.t. $d_G(v) = 1, vw \in E(G)$. Let $G' = G - v$.

Case 2.1.1. G does not have two vertices of maximum degree which are adjacent. So does G'.

By the induction hypothesis, we know that G' has a $(\Delta(G)+1) - AVDTCf$. If $d_G(w) \neq d_G(w_2)$ or $d_G(w) = d_G(w_2), C_f(w) \nsubseteq C_f(w_2)$, then we may obtain a $(\Delta(G)+1) - AVDTC$ of G easily. If $d_G(w) = d_G(w_2), C_f(w) \subseteq C_f(w_2)$, then let $l_2 \in \overline{C}_f(w_2), l_1 \in \{1, 2, \cdots, \Delta(G)+1\} - \{l_2, f(w)\}$. Define $f(vw) = l_2, f(v) = l_1$. Then the extended f is a $(\Delta(G)+1) - AVDTC$ of G.

Case 2.1.2. There exist two vertices of maximum degree in G which are adjacent.

Similar to Case 2.1.1 completely, we may obtain a $(\Delta(G)+2) - AVDTC$ of G from $(\Delta(G)+2) - AVDTC$ of G'.

Case 2.2. In G, all the vertices with degree at least 3 are of the same degree.

Case 2.2.1. There are vertex in C which is of degree 2 in G.

In this time, there exists $w, w_1, w_2 \in V(C)$, s.t. $ww_1, ww_2 \in E(G) \cap E(C), d_G(w_1) = 2, d_G(w) = \Delta(G) \geq 3$. Suppose $v \in V(G)$, s.t. $d_G(v) = 1, vw \in E(G)$. Let $G' = G - v$. We can obtain a $(\Delta(G)+1) - AVDTC$ of G from $(\Delta(G)+1) - AVDTC$ of G' or $(\Delta(G)+2) - AVDTC$ of G from $(\Delta(G)+2) - AVDTC$ of G' easily.

Case 2.2.2. $\forall u \in V(C)$, we have that $d_G(u) = \Delta(G) \geq 3$.

We know that $\chi_{at}(G) = \Delta(G) + 2$ by Lemma 2.1.

The proof is completed. □

3. The Adjacent-Vertex-Distinguishing Total Chromatic Number on the Square of Cycle

Let $C_n = v_1v_2\cdots v_nv_1$ be a cycle with order n. If positive integer $l > n$, then we provide that $v_l = v_r$, where $l \equiv r(\text{mod } n)$, and $r \in \{1, 2, \cdots, n\}$. We construct a new graph C_n^2, where $V(C_n^2) = V(C_n) = \{v_1, v_2, \cdots, v_n\}$, and $E(C_n^2) = \{v_iv_{i+1}|i = 1, 2, \cdots, n\} \cup \{v_iv_{i+2}|i = 1, 2, \cdots, n\}$. The new graph C_n^2 is called the square of cycle C_n. In this section, we give the adjacent-vertex-distinguishing total chromatic number of C_n^2.

Theorem 3.1. *Let C_n^2 be the square of the cycle C_n. Then*

$$\chi_{at}(C_n^2) = \begin{cases} 5, & n = 4; \\ 7, & n = 5; \\ 6, & n \geq 6. \end{cases}$$

Proof. If $n = 4$, then $C_n^2 = K_4$ (complete graph with order 4), so we know that $\chi_{at}(C_n^2) = 5$ (see [6]). If $n = 5$, then $C_n^2 = K_5$ (complete graph with order 5), so we know that $\chi_{at}(C_n^2) = 7$ (see [6]). We assume $n \geq 6$ in the following. Obviously $\chi_{at}(C_n^2) \geq 6$ by Lemma 1.3. We need only to prove that C_n^2 has a $6-AVDTC$ in the following. To this end, there are five cases to be considered.

Case 1. $n \equiv 0(\text{mod}3)$. We construct a mapping f from $V(C_n^2) \cup E(C_n^2)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

$$f(v_iv_{i+1}) \in \{1, 2, 3\}, \text{ and } f(v_iv_{i+1}) \equiv i(\text{mod}\, 3), i = 1, 2, \cdots, n;$$
$$f(v_j) \in \{1, 2, 3\}, \text{ and } f(v_j) \equiv j + 1(\text{mod}\, 3), j = 1, 2, \cdots, n;$$
$$f(v_jv_{j+2}) \in \{4, 5, 6\}, \text{ and } f(v_jv_{j+2}) \equiv j(\text{mod}\, 3), j = 1, 2, \cdots, n.$$

Obviously f is a 6-proper-total-coloring, and for $j = 1, 2, \cdots, n$, we have

$$f(v_j) = \{6\}, j \equiv 1(\text{mod}\, 3); \quad f(v_j) = \{4\}, j \equiv 2(\text{mod}\, 3)$$
$$f(v_j) = \{5\}, j \equiv 0(\text{mod}\, 3).$$

So f is $6 - AVDTC$.

Case 2. $n \equiv 1(\text{mod}\, 3)$. We construct a mapping f from $V(C_n^2) \cup E(C_n^2)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(v_i) \in \{1, 2, 3\}, f(v_i) \equiv i(\text{mod}\, 3), i = 1, 2, \cdots, n - 4;$$
$$f(v_{n-3}) = 4, f(v_{n-2}) = 2, f(v_{n-1}) = 3, f(v_n) = 4;$$
$$f(v_jv_{j+1}) \in \{1, 2, 3\}, f(v_jv_{j+1}) \equiv j - 1(\text{mod}\, 3), j = 2, 3, \cdots, n;$$
$$f(v_jv_{j+2}) = 6, j \equiv 1(\text{mod}\, 3), j = 1, 2, \cdots, n - 2;$$

$$f(v_jv_{j+2}) = 5, j \equiv 2(\bmod 3), j = 1, 2, \cdots, n-2;$$
$$f(v_jv_{j+2}) = 4, j \equiv 0(\bmod 3), j = 1, 2, \cdots, n-2;$$
$$f(v_{n-1}v_1) = 5, f(v_nv_2) = 6, f(v_1v_2) = 4.$$

Obviously, f is 6-proper-total-coloring. And

$$\overline{C}_f(v_1) = \{2\}, \quad \overline{C}_f(v_2) = \{3\}, \quad \overline{C}_f(v_{n-3}) = \{1\},$$
$$\overline{C}_f(v_{n-2}) = \{6\}, \quad \overline{C}_f(v_{n-1}) = \{4\}, \quad \overline{C}_f(v_n) = \{1\};$$

$$\overline{C}_f(v_j) = \begin{cases} \{5\}, & j \equiv 0(\bmod 3), \quad j = 3, 4, \cdots, n-4; \\ \{4\}, & j \equiv 1(\bmod 3), \quad j = 3, 4, \cdots, n-4; \\ \{6\}, & j \equiv 2(\bmod 3), \quad j = 3, 4, \cdots, n-4. \end{cases}$$

So f is a $6-AVDTC$.

Case 3. $n \equiv 2(\bmod 3)$, $n \geq 14$. We construct a mapping f from $V(C_n^2) \cup E(C_n^2)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(v_i) \in \{1, 2, 3\}, f(v_i) \equiv i(\bmod 3), i = 1, 2, \cdots, n-13;$$
$$f(v_{n-12}) = f(v_{n-9}) = f(v_{n-5}) = f(v_{n-2}) = 2$$
$$f(v_{n-11}) = f(v_{n-8}) = f(v_{n-4}) = f(v_{n-1}) = 3;$$
$$f(v_{n-10}) = f(v_{n-7}) = f(v_{n-3}) = f(v_n) = 4; f(v_{n-6}) = 1.$$
$$f(v_jv_{j+1}) \in \{1, 2, 3\}, f(v_jv_{j+1}) \equiv j-1(\bmod 3), j = 2, 3, \cdots, n-7;$$
$$f(v_1v_2) = f(v_{n-6}v_{n-5}) = 4; f(v_{n-5}v_{n-4}) = f(v_{n-2}v_{n-1}) = 1;$$
$$f(v_{n-4}v_{n-3}) = f(v_{n-1}v_n) = 2; f(v_{n-3}v_{n-2}) = f(v_nv_1) = 3;$$

$$f(v_jv_{j+2}) = \begin{cases} 4, & j \equiv 0(\bmod 3), j = 1, 2, \cdots, n-9; \\ 6, & j \equiv 1(\bmod 3), j = 1, 2, \cdots, n-9; \\ 5, & j \equiv 2(\bmod 3), j = 1, 2, \cdots, n-9; \end{cases}$$

$$f(v_{n-8}v_{n-6}) = f(v_{n-5}v_{n-3}) = f(v_{n-2}v_n) = f(v_{n-1}v_1) = 5;$$
$$f(v_{n-7}v_{n-5}) = f(v_{n-6}v_{n-4}) = f(v_{n-3}v_{n-1}) = f(v_nv_2) = 6;$$
$$f(v_{n-4}v_{n-2}) = 4.$$

Obviously, f is a 6-proper-total-coloring of C_n^2, and

$$\overline{C}_f(v_j) = \begin{cases} \{5\}, & j \equiv 0(\bmod 3), j = 3, 4, \cdots, n-12; \\ \{4\}, & j \equiv 1(\bmod 3), j = 3, 4, \cdots, n-12; \\ \{6\}, & j \equiv 2(\bmod 3), j = 3, 4, \cdots, n-12; \end{cases}$$

$$\overline{C}_f(v_{n-11}) = \overline{C}_f(v_{n-4}) = \{5\}$$
$$\overline{C}_f(v_{n-10}) = \overline{C}_f(v_{n-7}) = \overline{C}_f(v_{n-3}) = \overline{C}_f(v_n) = \{1\};$$
$$\overline{C}_f(v_{n-9}) = \overline{C}_f(v_{n-2}) = \{6\}, \quad \overline{C}_f(v_{n-8}) = \overline{C}_f(v_{n-1}) = \{4\};$$
$$\overline{C}_f(v_{n-6}) = \overline{C}_f(v_1) = \{2\}; \quad \overline{C}_f(v_{n-5}) = \overline{C}_f(v_2) = \{3\}.$$

By careful examination, we know that f is a $6-AVDTC$ of C_n^2.

Case 4. $n = 11$. We construct a mapping f from $V(C_n^2) \cup E(C_n^2)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(v_1) = f(v_5) = 1; \quad f(v_2) = f(v_6) = f(v_9) = 2;$$
$$f(v_3) = f(v_7) = f(v_{10}) = 3;$$
$$f(v_4) = f(v_8) = f(v_{11}) = 4; \quad f(v_1v_2) = f(v_5v_6) = 4;$$
$$f(v_2v_3) = f(v_6v_7) = f(v_9v_{10}) = 1;$$
$$f(v_3v_4) = f(v_7v_8) = f(v_{10}v_{11}) = 2; \quad f(v_4v_5) = f(v_8v_9) = f(v_{11}v_1) = 3;$$
$$f(v_1v_3) = f(v_4v_6) = f(v_5v_7) = f(v_8v_{10}) = f(v_{11}v_2) = 6;$$
$$f(v_2v_4) = f(v_3v_5) = f(v_6v_8) = f(v_9v_{11}) = f(v_{10}v_1) = 5; \quad f(v_7v_9) = 4.$$

By easy verification, we know that f is 6-proper-total-coloring. And

$$\overline{C}_f(v_1) = \overline{C}_f(v_5) = \{2\}; \quad \overline{C}_f(v_2) = \overline{C}_f(v_6) = \{3\};$$
$$\overline{C}_f(v_3) = \overline{C}_f(v_{10}) = \{4\};$$
$$\overline{C}_f(v_4) = \overline{C}_f(v_8) = \overline{C}_f(v_{11}) = \{1\};$$
$$\overline{C}_f(v_7) = \{5\}; \quad \overline{C}_f(v_9) = \{6\}.$$

Again by careful examination, we can obtain that f is a $6-AVDTC$ of C_{11}^2.

Case 5. $n = 8$.

We construct a mapping f from $V(C_n^2) \cup E(C_n^2)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(v_i) \in \{1, 2, 3, 4\}, f(v_i) \equiv i (\bmod 4), i = 1, 2, \cdots, 8;$$
$$f(v_iv_{i+1}) \in \{1, 2, 3, 4\}, \text{ and } f(v_iv_{i+1}) \equiv i - 1 (\bmod 4);$$

$$f(v_jv_{j+2}) = \begin{cases} 5, & j \equiv 1, 2 (\bmod 4), \quad j = 1, 2, \cdots, 8; \\ 6, & j \equiv 3, 0 (\bmod 4), \quad j = 1, 2, \cdots, 8. \end{cases}$$

Obviously, f is a 6-proper-total-coloring of C_n^2, and

$$\overline{C}_f(v_1) = \overline{C}_f(v_5) = \{2\}; \quad \overline{C}_f(v_2) = \overline{C}_f(v_6) = \{3\};$$
$$\overline{C}_f(v_3) = \overline{C}_f(v_7) = \{4\}; \overline{C}_f(v_4) = \overline{C}_f(v_8) = \{1\}.$$

So f is a $6-AVDTC$ of C_8^2.

The proof is completed. □

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