International Journal of Pure and Applied Mathematics
Volume 19 No. 1 2005, 107-110

WEIERSTRASS SETS AND RAMIFICATION POINTS OF LINE BUNDLES ON CURVES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a smooth curve of genus g, $L \in \mathrm{Pic}(X)$ and $(P_1, \dots, P_n) \in X$ with $P_i \neq P_j$ for all $i \neq j$. $(P_1, \dots, P_n)$ is a Weierstrass (resp. strict Weierstrass) n-ple for L if there are integers $a_i \geq 0$ (resp. $a_i > 0$) such that $\sum_{i=1}^{n} a_i \leq h^0(X, L)$ and $h^0(X, L(-a_1P_1 - \cdots - P_n)) > h^0(X, L) - \sum_{i=1}^{n} a_i$. Here we study n-ples which are Weierstrass n-ples for large classes of line bundles on X, mainly when $n \leq 2$.

AMS Subject Classification: 14H55
Key Words: Weierstrass point, Weierstrass n-ple, Weierstrass pair, gap sequence, ramification point

1. Weierstrass Sets for Line Bundles on Curves

Let X be a smooth and connected curve of genus g, $P \in X$ and $L \in \mathrm{Pic}^d(X)$ such that $h^0(X, L) > 0$. We will say that P is a Weierstrass point of L or a ramification point of L if $h^0(X, L(-tP)) > 0$, where $t = h^0(X, L)$. We want to study the same concept for n-ples of points of X ([1], Definition 4.1). It is widely believed that n-pointed curves are interesting objects that deserve to be studied for their own sake. Furthermore, in the theory of generalized linear series do to D. Eisenbud and J. Harris 1-pointed and n-pointed curves appear as a tool to study smooth curves without any marked point ([5], [6], [11]).

Received: January 31, 2005

Definition 1. Fix an integer $n \geq 1$, $L \in \mathrm{Pic}(X)$ and n distinct points $P_1, \dots, P_n$ of X. Set $w(L; P_1, \dots P_n) := \sum(\alpha_1 + \cdots + \alpha_n + h^0(X, L(-\alpha_1 P_1 - \cdots - \alpha_n P_n)) - h^0(X, L))$, where the sum is over all non-negative integers $\alpha_1, \dots, \alpha_n$ such that $h^0(X, L(-\alpha_1 P_1 - \cdots - \alpha_n P_n)) > 0$. Set $w(L; P_1, \dots P_n)_- := \sum(\alpha_1 + \cdots - \alpha_n + h^0(X, L(-\alpha_1 P_1 - \cdots - \alpha_n P_n)) - h^0(X, L))$, where the sum is over all integers $\alpha_1 > 0, \dots, \alpha_n > 0$ such that $h^0(X, L(-\alpha_1 P_1 - \cdots - \alpha_n P_n)) > 0$. Call the non-negative integer $w(L; P_1, \dots P_n)$ (resp. $w(L; P_1, \dots P_n)_-$) the weight (resp. the strict weight) of the n-ple $(P_1, \cdots, P_n)$. We will say that $(P_1, \dots, P_n)$ is a Weierstrass (resp. strict Wierstrass) n-ple for L if $w(L; P_1, \dots, P_n) > 0$ (resp. $w(L; P_1, \dots P_n)_- > 0$).

We work over an algebraically closed field $\mathbb{K}$. Even in the case $n = 1$ and $L = \omega_X$ the above definition of weight and of Weierstrass n-ple is the best one only if $\mathrm{char}(\mathbb{K}) = 0$ ([7], [8] and [9]). Nevertheless, even in positive characteristic it gives certain informations and hence we will state all the characteristic-free results in full generality.

Here we list our results.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *Let X be a bielliptic curve of genus $g \geq 4$, $u : X \to C$ its associated double covering with C an elliptic curve and $P, P_1, P_2 \in X$. There are exactly $2g - 2$ points of X at which u is not étale. If $L \in \mathrm{Pic}^d(X)$ is base point free and $h^1(X, L) \geq 2$, then d is even and there is $A \in \mathrm{Pic}^{d/2}(C)$ such that $u^*(A) \cong L$ and $h^0(X, L) = h^0(C, A)$. For any $A \in \mathrm{Pic}^x(C)$ with $x \leq g - 2$ we have $h^0(X, u^*(A)) = h^0(C, A)$ and hence P is a base point of $u^*(A)$ if and only if $u(P)$ is a base point of A; furthermore, if $P \in X$ not in the base locus of $u^*(A)$, then it is a Weierstrass point for $u^*(A)$ if and only if either $u(P)$ is a Weierstrass point of A or u is not étale at P.*

(a) P is a Weierstrass point of a degree 4 spanned line bundle on X if and only if u is not étale at P.

(b) Assume $P_1 \neq P_2$ and $u(P_1) = u(P_2)$. The pair (P_1, P_2) is a strict Weierstrass pair of the spanned line bundle $\mathcal{O}_X(2P_1 + 2P_2)$. There is a base point free $L \in \mathrm{Pic}^g(X)$ such that $h^1(X, L) = 1$, $h^0(X, L) \geq 2$ and (P_1, P_2) is not a Weierstrass pair of L.

Proposition 1. *Let X be a smooth and connected curve of genus $g \geq 4$ and $P \in X$. P is a Weierstrass point of all base point free special line bundles on X if and only if X is hyperelliptic and P is a Weierstrass point of X, i.e. a Weierstrass point for ω_X.*

Proposition 2. *Let X be a smooth and connected curve of genus $g \geq 4$ and $P, Q \in X$ such that $P \neq Q$. The pair (P, Q) is a Weierstrass pair of all* *base point free special line bundles on X if and only if X is hyperelliptic and either P is a Weierstrass point of X or Q is Weierstrass point of X or Q is the image of P by the hyperelliptic involution of X.*

Proposition 3. *Fix integers g, n such that $2 \le n \le g-2$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \ge 2g-1$. Let X be a smooth curve of genus g and $P_i \in X$, $1 \le i \le n$, such that $P_i \ne P_j$ for all $i \ne j$. Assume that $(P_1, \dots, P_n)$ is a strict Weierstrass n-ple for all $L \in \mathrm{Pic}(X)$ such that $h^1(X, L) > 0$ and $h^0(X, L) \ge n+1$. Then there exists an index i such that $1 \le i \le n$ and $h^0(X, \mathcal{O}_X(P_i + \sum_{i=1}^n P_i)) \ge 2$.*

Proof of Propositions 1, 2, 3 and of the last sentence of Theorem 1. Since the hyperellitic case is easy, we assume X not hyperelliptic. Let $\phi : X \to \mathbf{P}^{g-1}$ be the canonical embedding of X. Fix $P \in X$ and let T be the tangent line of $\phi(X)$ at $\phi(P)$. Since $\phi(X)$ spans $\mathbf{P}^{g-1}$ there are $Q_1, \dots, Q_{g-2} \in X$ such that $\mathbf{P}^{g-1}$ is spanned by $T \cup \{\phi(Q_1), \dots, \phi(Q_{g-2})\}$. Let $B \ge 0$ be the base locus of the line bundle $\omega_X(-Q_1 - \cdots - Q_{g-2})$. By construction $h^0(X, L) = 2$ and $h^0(X, L(-2P)) = 0$, proving Proposition 1. The other results are proved in a similar way. We only need to remark that the assumptions on $\mathrm{char}(\mathbb{K})$ made in the statement of Proposition 3 imply that the gap sequence of $\phi(X)$ at its general point is the ordinary one and by [10] a general hyperplane section of $\phi(X)$ is in linearly general position. □

Proof of Theorem 1. Since $g \ge 4$ and $\mathrm{char}(\mathbb{K}) \ne 2$ the ramification divisor $R \subset C$ has degree $2g-2 > 2$ (Riemann-Hurwitz). Furthermore, the pair (X, u) is uniquely determined by C, R and the choice of $R_1 \in \mathrm{Pic}^{g-1}(C)$ such that $R_1^{\otimes 2} \cong \mathcal{O}_C(R)$. The $2g-2$ points in $u^{-1}(R)$ are the points of X at which u is not étale. By the projection formula for any $A \in \mathrm{Pic}(C)$ we have $h^0(X, u^*(A)) = h^0(C, A) + h^0(C, A \otimes R_1^*)$ and hence $h^0(X, u^*(A)) = h^0(C, A)$ if $\deg(A) \le g-2$. Let $\phi : X \to \mathbf{P}^{g-1}$ be the canonical embedding. Set $Y := \phi(X)$. By [2], Theorem 2.1, there is $O \in \mathbf{P}^{g-1} \backslash Y$ such that the linear projection $m : \mathbf{P}^{g-1} \backslash \{O\} \to \mathbf{P}^{g-2}$ induces a two-to-one morphism from Y onto a linear normal elliptic curve $C' \cong C$, and, up to this isomorphism, $m|Y = u$ and in particular $m|Y$ is not étale at $\phi(P)$ if and only if u is not étale at P. Furthermore, Y is contained in the degree $g-1$ cone S with vertex O and base C'. Let L be a base point free line bundle on X. If $1 \le \deg(X) \le g-1$ it is easy to check using Castelnuovo-Severi inequality (or see [3], Section 2, or [4], Example 1.13) the existence of $A \in \mathrm{Pic}(C)$ such that $L \cong u^*(A)$ and $h^0(X, L) = h^0(C, A)$. The same is true if $h^1(X, L) \ge 2$ by [4], part (v) of Proposition 2.2. Everything follow from the cohomology of line bundles on C. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, S.J. Kim, Weierstrass multiple loci of n-pointed algebraic curves, *J. Algebra*, **199**, No. 2 (1998), 455-471.

[2] A. Del Centina, Remarks on curves admitting an involution of genus ≥ 1, *Boll. U.M.I.*, **4-B**, No. 6 (1985), 671-683.

[3] M. Coppens, C. Keem, G. Martens, Primitive linear series on curves, *Manuscripta Math.*, **77** (1992), 237-264.

[4] M. Coppens, G. Martens, Linear series on 4-gonal curves, *Math. Nachr.*, **213** (2000), 35-55.

[5] D. Eisenbud, J. Harris, Limit linear series: basic theory, *Invent. Math.*, **85** (1986), 337-371.

[6] D. Eisenbud, J. Harris, Existence, decomposition and limits of certain Weierstrass points, *Invent. Math.*, **87** (1987), 495-515.

[7] M. Homma, Linear systems on curves with no Weierstrass points, *Bol. Soc. Brasil Mat.*, **23** (1992), 93-108.

[8] D. Laksov, Weierstrass points on curves, In: *Tableaux de Young et Foncteurs de Schur en Algebre et Géometrie*, Astérisque, **87/88** (1981), 221-247.

[9] D. Laksov, Wronskians and Plücker formulas for linear systems on curves, *Ann. Scient. École Norm. Sup.*, **17**, No. 4 (1984), 565-579.

[10] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.

[11] G. Welters, A theorem of Gieseker-Petri type for Prym varieties, *Ann. Scient. École Norm. Sup.*, **18**, No. 4 (1985), 671-683.