

SCALE FREE $SL(2,R)$ ANALYSIS AND
THE PICARD'S EXISTENCE AND
UNIQUENESS THEOREM

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Abstract: The existence of higher derivative discontinuous solutions to a first order ordinary differential equation is shown to reveal a nonlinear $SL(2,R)$ structure of analysis in the sense that a real variable t can now accomplish changes not only by linear translations $t \rightarrow t+h$ but also by inversions $t \rightarrow 1/t$. We show that the real number set has the structure of a positive Lebesgue measure Cantor set. We also present an extension of the Picard's Theorem in this new light.

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1. Introduction

The most basic ingredient of real analysis is of course a variable, living in the real number line, which is assumed to undergo changes by linear translations. The framework of ordinary analysis is therefore essentially *linear*, the relevant concepts of limit, differentiability, etc. being formally defined to satisfy the simple transformation properties of the linear translation group. Notice, for example, that the limit $t \rightarrow t_0$ means the linear, continuous decrease (as an action of the translation group) of the distance $|t - t_0|$ to zero and so on. The application of this linear analysis in more complex dynamical problems, for in-

stance, necessitates explicit breaking of this linear structure by incorporating nonlinear interactions (terms) in the underlying differential (and/or integral) equations. The recent investigations, however, reveals the surprising existence of some *nonlinear dynamical structures* right at the heart of the ordinary analysis [2, 3, 4, 5], thereby extending the framework of the linear ordinary analysis to the scale free $SL(2, \mathbf{R})$ analysis. This extension is based on the realization that the simplest linear, scale free, differential equation, viz.,

$$t \frac{d\tau}{dt} = \tau, \quad \tau(1) = 1 \quad (1)$$

admits a generalized class of nonlinear solutions of the form $\tau_g = \tau_s(1 + O(\eta^{2^n}))$, having discontinuities in the 2^n -th order derivatives at $t = 1$ [5]. Here, $\tau_s = t$ is the standard solution and $t \rightarrow t/t_0 = 1 + \eta$ denotes a rescaled variable close to 1. The discontinuity in the 2^n -th order derivative stems from the rescaling invariance of equation (1) which allows one to inject an infinite sequence of arbitrary, *infinitesimal* scale factors into the standard solution τ_s . Consequently, the small scale variable η is raised to a random *infinitesimal* variable, residing in a nonstandard real line \mathbf{R} [7], rather than in the ordinary real line R . Indeed, let $t_{\pm} = 1 \pm \eta$, $\eta > 0$. Then obviously $\tau_{s-} = \tau_s(t_-) = 1 - \eta$. Now, it is easy to verify that this exact solution can as well be derived starting from an initial approximation and then obtaining self similar (multiplicative) correction factors recursively, viz., $\tau_s = (1/t_+) \tau_1$ when the correction factor satisfies the self similar equation

$$t_{1-} \frac{d\tau_{1-}}{dt_{1-}} = \tau_{1-} \quad (2)$$

in the smaller scale variable $t_{1\pm} = 1 \pm \eta^2$ in the neighbourhood of 1, and so on, since $t_- = \frac{1}{(1+\eta)(1+\eta^2)(1+\eta^4)\dots}$. The freedom of a residual¹ rescaling in equation (2) of the form $\alpha_0 t_{1-} = t'_{1-} = 1 - \eta'$, $\eta' = \alpha_0(\eta^2 - \epsilon_0/\alpha_0)$, $\alpha_0 = 1 + \epsilon_0$, ϵ_0 being an arbitrarily small real number drawn at random from the interval $\eta^2 \leq \epsilon_0 \ll \eta$, however, induces a small scale modulations (fluctuations) in t'_{1-} and so on to the higher order iterates t'_{n-} generated self similarly from equation (2) over smaller and smaller scales. The new, nontrivial solution [5] is thus obtained as

$$\tau_N = C \prod \frac{1}{t'_{n+}}, \quad C = \prod t'_{n+}(0). \quad (3)$$

¹The said rescaling is residual in the following sense. The trivial rescaling $t \rightarrow t/t_0$ shifts a variable near t_0 to one near 1. The present rescaling reveals, the so far hidden, freedom of injecting a small scale uncertainty in the neighbourhood of a point.

Clearly, the smaller scales reveals a nonlinear structure in the neighbourhood of a real number. Notice, for example, that $\eta'_n = \alpha_n(\alpha_{n-1}^2 \eta_{n-1}'^2 - \epsilon_n/\alpha_n)$, ϵ_n representing higher order uncertainties and $\alpha_n t_{n+} \neq t'_{n+}$. The second inequality tells the nontriviality of the residual rescaling symmetry, which is responsible for the higher derivative discontinuity of the new solution τ_g at $t = 1$, provided, of course, the rescaling is performed an infinite number of times (for a proof see Appendix). Notice also that τ_g is 2-nd derivative discontinuous if the nontrivial rescalings are injected at the 2-nd and an infinite sequence of subsequent iterates, 4-th derivative discontinuous when the freedom of this rescaling is used for the first time at the 3-rd, instead of 2-nd, iterate and so on. In general, the new solution would have discontinuous 2^{n-1} derivative if the nontrivial rescaling is utilized at the n -th iterates (the standard solution τ_s is recovered when this rescaling is postponed till the infinite number of iterations [5] (c.f. Appendix)).

In view of this class of discontinuous solutions to the simplest scale invariant linear equation (1), the ordinary real number system R needs to be extended, accommodating this small scale nondifferentiable structure, to a nonstandard – like real number system \mathbf{R} . Notice that the generalized solution of equation (1) can indeed be written as $\tau_g = t(1 \pm \epsilon\phi(t_1))$, $\phi(t) = \tau_N(t)/t$, $t_1 = \epsilon t$, where $\frac{d\phi}{d \ln t_1} = 0$, τ_N being the new solution (3). Consequently, every ordinary real number, for instance, t_0 gets extended to a set of the form $(t_0)_\epsilon = \{t : t_0 - \epsilon < t < t_0 + \epsilon, \epsilon \neq 0, O(\epsilon^2) = 0, t \neq t_0\} \subset \mathbf{R}$, where ϵ denotes the irreducible uncertainty even when the real number is evaluated with an “infinitely precise measurement”. Clearly, the inclusion $\dots (t_0)_{\epsilon^2} \subset (t_0)_\epsilon$ is satisfied for higher precision evaluation of t_0 . We note that the generalized solution, as it is obtained in [5], is constructed purely as a function defined in R and hence the extended set \mathbf{R} constructed out of the generalized solution coincides exactly with R viz., $\mathbf{R} = R$. We remark that this feature of our extension distinguishes it from the work of Robinson [7]. In the traditional approach of nonstandard analysis, infinitesimals are defined as an equivalence class of sequences of real numbers. As a result, infinitesimals, being constructed out of real numbers, can be thought of as *extraneous* to the set of ordinary real number set. Our results, on the other hand, suggest that *infinitesimals are indeed members of the real number set R* and so might instead be defined formally even in the real number set R . This approach is being developed recently by Bose [1]. In view of this existence of infinitesimals in R , with random properties, the new solution $\tau_g \in \mathbf{R}$ can as well be interpreted to reveal a new mode of change that is available to a (real) variable t , viz., t can change near an infinitesimal neighbourhood of $t = 1$ by (random) inversions: $t_- \rightarrow t_-^{-1} = t_+$, $t_\pm = 1 \pm \eta$, for an infinitesimal η , defined

by $\eta \neq 0$ but $O(\eta^2) = 0$, besides the ordinary translations over the ordinary real number line. Notice that the scale free equation (1) is invariant under the transformation $t \rightarrow 1/t$, $\tau \rightarrow 1/\tau$ and so $t_- \rightarrow t_-^{-1} = t_+$ indeed stands for the generalized solutions considered here. The possibility of (random) inversions breaks the exact determinacy of a real variable. Further, the scale invariance tells that the small scale fluctuating behaviour of a real variable (over η) is reproduced self similarly over the smaller nonlinear scales (η^{2^n}), though the self similarity is respected in a statistical sense. Accordingly, the linear framework of the ordinary analysis gets extended to a $SL(2, \mathbf{R})$ analysis. One therefore infers that the real number system R , as it is ordinarily understood in the context of the linear (ordinary) analysis, enjoys a host of richer “dynamical” properties which get revealed in the $SL(2, \mathbf{R})$ analysis. We have already discussed some of these new features of this analysis in [4]. Here, we present an analysis of the Picard’s theorem [8] in the light of the new generalized solutions of equation (1). To this end we first discuss how the arguments used in the simple equation (1) can be extended to more general equations (Section 2). Next we discuss some salient features of the computational model introduced in [5] and also show that \mathbf{R} has the structure of a positive Lebesgue measure Cantor set [6]. The (box counting) dimension of a point in \mathbf{R} (hence in R) has the value ν , where $\nu = \frac{\sqrt{5}-1}{2}$ is the golden mean (Section 3). Before stating the Picard’s Theorem in this new light, we present an extension of the ordinary Riemann integration on \mathbf{R} (Section 4).

2. General Case

To prepare for a discussion of the Picard’s Theorem, let us clarify further the set up of the scale free $SL(2, \mathbf{R})$ analysis. To this end, we consider the more general linear ODE of the form

$$\frac{d \ln \tau}{d \ln t} = f(t, \tau), \quad \tau(1) = \tau_0. \quad (4)$$

Let $\tilde{\tau}(t)$ be the corresponding standard solution. This solution is exactly determinable, in principle, in the linear analysis. In the extended framework it, however, corresponds to the zeroth order solution only. Based on this solution, one can, however, generate a more *complex* solution in the following way. Following the steps outlined in Section 1, let us write the generalized solution close to $t = 1$ (more precisely in a neighbourhood of 1) as $\tau_{g-}(t_{g-}) \equiv \tau(t_-) = \tilde{\tau}(1/t_+)\tau'_-$. Then it is easy to verify that τ'_- would satisfy

an equation of the form

$$\frac{d \ln \tau'_-}{d \ln t_{1-}} = f'(t_{1-}, \tau'_-) \tag{5}$$

on the smaller nonlinear scale $\ln(1 - \eta^2)$ where

$$f'(t_{1-}, \tau'_-) = \frac{(t_+ f(t_-, \tau_-) - t_- f(1/t_+, \tilde{\tau}(1/t_+)))}{t_+ - t_-}.$$

Clearly, the generalized solution τ_g would belong to the higher order nondifferentiable classes of solutions when the infinite sequence of nontrivially rescaled (equivalently, random) variables t'_{n-} are incorporated as indicated above.

Notice that the exact self similarity on the smaller (nonrandom) scales $\ln(1 - \eta^2)$, etc. is obtained only for the simplest equation (1). For, a t -dependence either of the form $f(t)$ or $f(t, x)$ breaks this exact self similarity. Nevertheless, an approximate self similarity is maintained on scales *randomised* by the nonlinear t -dependence, besides the above mentioned freedom of rescaling, when the random scale is defined, for instance, by $t'_{1-} = t_1^\mu \approx 1 - \mu\eta^2$, where $\mu = f'(t_{1-}, \tau'_-)/f(t_{1-}, \tau'_-) \approx 1$, which remains almost constant over the scale η^2 . Note that, randomisation of scales by explicit nonlinearity in the ODE already simulates the residual rescaling freedom of equation (1). The emergence of random behaviour in dynamical systems (for instance, one dimensional maps, higher order nonlinear ODEs, etc.) from truncation errors are well known in deterministic chaos. Here, we point out equivalent behaviour even in one dimensional *linear* ODEs as an effect of infinitesimal nonlinear scales (elements) in real number system. As examples we consider two simple equations in the following.

Case 1. (Modulated Exponential) Small scale modulations in the ordinary exponential function, defined by

$$\frac{d\tau}{dt} = \tau, \tag{6}$$

is expected in the present formalism. One verifies that the generalized solution near $t = 1$ is obtained as

$$\exp_g(t_-) = e^{1/t_+ + 1/t_{1+} + \dots}, \tag{7}$$

where $t_{1+} = 1 + (1/t_+ - t_-) = 1 + \frac{\eta^2}{1-\eta}$, so that $\mu = 1/(1 - \eta)$, and the equality in (7) is up to a multiplicative factor. In a computational problem (truncated) nonlinear terms in μ would build up over time to introduce small scale *random*

oscillations in the ordinary exponential. Such behaviour would of course be common to *any* deterministic functions of linear analysis.

Case 2. (Quadratic Nonlinearity) Let us consider the equation

$$t \frac{d\tau}{dt} = \tau^2, \quad \tau(1) = 1 \quad (8)$$

having the exact solution $\tau_0 = 1/(1 - \ln t)$. Writing $\tau_g(t_{g-}) \equiv \tau_0(t_-) = \tau_0(1/t_+) \tau_{1-}$, as usual, we get

$$\frac{d \ln \tau_{1-}}{d \ln t_{1-}} = \mu \tau_{1-}, \quad (9)$$

where

$$\mu = \frac{t_+ - t_- \left(\frac{\tau_0(1/t_+)}{\tau_0(t_-)} \right) (1 - \ln t'_{1-})}{t_+ - t_- (1 - \ln t_-)} \approx 1.$$

We point out that an equation of the form (9) is truly nonlinear in the sense that nonlinear random scale $t'_{1-} = t''_{1-} \approx 1 - \mu \eta^2$ is itself determined by the “unknown” function τ_{1-} through μ and hence does not lead to an exact solution in the ordinary sense.

To conclude this section, we remark that the infinitesimal fluctuations induced to the solutions to equation (4) because of $SL(2, R)$ inversions would have a universal feature determined by the functional dependence $f(t, \tau)$ close to $t = 1$.

3. Infinitesimals, Computation and Cantor Sets

In ref [5] we presented a computational model of infinitesimally small real numbers. Here we explain the relationship of inversions and infinitesimally small numbers in the context of this model. Recall that every ordinary real number t is replaced by $t_\epsilon = t(1 \pm \epsilon \phi(t_1))$ so that t_ϵ essentially corresponds to a continuum set $(t)_\epsilon$. Now in a computational problem, a real number is only treated as a finite precision (decimal/binary) representation. Consequently, the real number set is covered by a countable collection of disjoint open intervals of the form t_ϵ , where ϵ now denotes the finite precision with which the real numbers are evaluated (calculated). Notice that ϵ can have values, in decimal representation, of the form 0.5 (for integers, so that 1, for instance, stands actually for the open interval (0.5, 1.5) and so on), 0.05, 0.005, 0.0005, ... As a result, in any computation the real number set, although conceived as a connected,

continuum set (without any gap), is actually realized as a totally disconnected, countable set (equivalent to the set of rationals). As one improves upon the accuracy and approaches to the infinite precision the connected continuum set of linear analysis is thought to have been recovered only in the limit. In view of the scale free extension, one, however, meets with an obstacle in the form of *irreducible infinitesimal uncertainties*, thereby realizing the continuum structure of the real number set but for the total disconnectedness. In fact, the extended real number set acquires the structure of a Cantor set with a positive Lebesgue measure [6].

Lemma. *The set \mathbf{R} has the box counting dimension σ , $\sigma = 1 + \ln(1 + \epsilon\phi)/\ln t$. Further, σ has the asymptotic limits $\lim_{t \rightarrow 0^+} \sigma = \nu$ and $\lim_{t \rightarrow \infty} \sigma = 1 + \nu$ where $\nu = \frac{\sqrt{5}-1}{2}$.*

Proof. In view of the structure of new solution presented in Section 1, \mathbf{R} is an extension of R in the following sense. Let $N_{0\delta}$ be the number of open balls (intervals) of radius δ covering an open interval of R . The box counting dimension of R is $\lim_{\delta \rightarrow 0^+} \frac{\ln N_{0\delta}}{\ln \delta^{-1}}$ which, in fact, equals 1. To cover \mathbf{R} the set $\{N_{0\delta}\}$, however, is not sufficient. One, in fact, needs to subdivide each of subintervals of size δ into smaller subdivisions of size δ^2 and this process of subdivisions is continued recursively over smaller and smaller scales $\delta^4, \delta^8, \dots$. In each step \mathbf{R} is covered better but for a tiny residual part, which could only be measured (detected) looking at the next level of finer scale. Accordingly, these subdivisions over finer scales facilitates one to measure the required “length” of \mathbf{R} more and more accurately only in a progressive manner.² Let, in the first step of subdivisions, the fractional increase in the number of subdivisions be $N_{1\delta^2}/N_{0\delta} (= \lambda < 1)$, where $N_{1\delta^2}$ is the number of subdivisions of scale δ^2 needed to cover the boundary “points” of the original interval. Notice that the interior points in the open interval concerned are already covered well by the zeroth level covering balls and so finer scale covering balls are irrelevant for these points. Continuing this process sequentially over smaller scales (as indicated above) one then gets, for instance, as the fractional increase $N_{2\delta^4}/N_{1\delta^2}$ and so on. Because of the (relative) self similarity of the subdivisions one concludes $N_{2\delta^4}/N_{1\delta^2} = N_{1\delta^2}/N_{0\delta}$, etc. Consequently, total number of subdivisions of different sizes needed to cover the said interval of \mathbf{R} is $\tilde{N} = \sum N_{n\delta^{2^n}} = N_{0\delta}(1 + \lambda + \lambda^2 + \dots)$. Hence the box counting dimension of \mathbf{R} , evaluated relative to scale δ , is obtained as $\sigma = \lim_{\delta \rightarrow 0^+} \frac{\ln \tilde{N}}{\ln \delta} = 1 + \frac{\lambda}{1-\lambda}$.

Let $\mathbf{t} \in \mathbf{R}$. Then the “length” of the open interval $(\mathbf{0}, \mathbf{t})$ would be given by

²The concept of a perfect (or total) covering (and/ or infinite precision) is, however, illusory. What is more meaningful is the continual progress towards perfection [2, 3].

t^σ , where $t \in R$. Again, in view of the new solution we write $\mathbf{t} = t(1 + \epsilon\phi)$. Comparing the two we deduce the first result. The value of λ is thus determined by the nontrivial factor in \mathbf{t} . Its exact value is, however, unimportant for our purpose. That it is nonzero is actually more significant.

To prove the remaining part of the lemma, we note that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \sigma &= \lim_{t^{-1} \rightarrow \infty} (1 - \epsilon(\frac{t^{-1}}{\ln t^{-1}})\tau(t_1) + \text{higher order terms}) \\ &= \lim_{t^{-1} \rightarrow \infty} (1 - \epsilon(\frac{t^{-1}}{\ln t^{-1}})\tau(t_1) + \epsilon^2(\frac{t^{-2}}{\ln t^{-2}})(\tau(t_1))^2 + \dots) \end{aligned}$$

(τ denotes the new solution of (3)). As t^{-1} approaches to ∞ through higher and higher order scales, there exists a sufficiently large interval in R , when $\pi(t^{-1}) = \frac{t^{-1}}{\ln t^{-1}} \approx t^{-1} \approx \epsilon^{-1}$. Accordingly, in this interval $\lim_{t \rightarrow 0^+} \sigma = \lim_{t^{-1} \rightarrow \infty} (1/(1 + \tau(t_1))) = \lim_{t^{-1} \rightarrow \infty} (1/(1 + (1/(1 + \epsilon^2(\frac{t^{-1}}{\ln t^{-1}})\tau(t_2))))$, where $t_2 = \epsilon^2 t$, and so on. Notice that the r.h.s of the first equality mimics exactly an application of inversion on the first two terms of the logarithmic expansion of σ . The desired limit now follows from the continued fraction expansion of the golden mean ν . Similar arguments also hold for the other limit when one makes use instead $\phi(t) = t\tau(t_1^{-1})$, $\tau(t_1^{-1}) = 1/\tau(t_1)$ [3]. \square

Remark 1. To see the origin of inversion, let us reconsider the computational model introduced above. Let a computation could only distinguish integer numbers, so that $1 \equiv (1)_{.5}$, for instance. Consequently, $(1)_\epsilon = 1 + \epsilon\mu$, $\mu = \mu_- \cup \mu_+$, $\mu_- = (-1, 0)$, $\mu_+ = (0, 1)$, $\epsilon = 0.5$. The number 1, therefore, corresponds to *any* number drawn perfectly at *random* from $(0.5, 1.5)$ (the assumption of randomness is codified in the residual rescaling symmetry c.f. Section 1). This is realized in the above by letting ϵ instead to lie in the interval $(-0.5, 0) \cup (0, 0.5)$ and $\mu = \phi \sim O(1)$. The irreducible fluctuations as encoded in the new solution ϕ now tells that a point in $(-0.5, 0)$ could fluctuate to a point in $(0, 0.5)$ (by inversion, viz., by scaling and flipping of sign) and vice versa, instead of pure translations, as assumed in ordinary analysis. Further, the especial role of inversions in a Cantor set is also intuitively clear because a point in a Cantor set can change to another only by an inversion (i.e., by a discrete jump) because of the gaps (voids) separating the Cantor points [4].

Remark 2. To visualize the Cantor set like structure of $(0)_\epsilon$, we note that $(0)_\epsilon = (-\epsilon, 0_0) \cup (0)_{\epsilon^2} \cup (0_0, \epsilon)$, where 0_0 denotes the zero (0) at level 0, and so on recursively over smaller scales. Accordingly, in this extended framework, the number 0, for instance, is represented by the set $\cup_0^\infty \{(-\epsilon^{2^n}, 0_n) \cup (0)_{\epsilon^{2^{n+1}}} \cup (0_n, \epsilon^{2^n})\}$, 0_n being the n -th level zero. Because of the natural in-

clusion $0_n \supset 0_{n+1}$, the set $\{0_n\}$ represents a finer and finer realization of the originally coarse grained zero (0) as the accuracy of the computation is increased. In linear analysis, however, these fine structures are ignored so that the ordinary singleton $\{0\}$ is reproduced. Notice that the usual singletons $\{t_0\}$ in the reduced (coarse grained) set R are reinserted once the nontrivial SL(2,R) generator is frozen.

Remark 3. The significance of $\pi(t^{-1})$ in the prime number theorem and other number theoretic results will be considered elsewhere.

4. The Picard's Theorem

In view of the Cantor set like structure of \mathbf{R} (every point of R is actually got replaced by a Cantor set, c.f., Remark 2), the definition of the ordinary (Riemann) integration needs to be extended. Notice that the Cantor set of $(1)_\epsilon$, for instance, is written as $(1)_\epsilon = \cup_0^\infty [(1_n - \epsilon^{2^n}, 1_n + \epsilon^{2^n}) - \{1_n\}]$ (c.f., Remark 2) of length $2 \sum_0^\infty [\epsilon^{2^n} - (\epsilon^{2^n} - \epsilon^{2^{n+1}})] = 2 \sum_1^\infty \epsilon^{2^n}$ which is essentially zero, being of higher order infinitesimal, relative to the first order infinitesimal ϵ . Notice that the length of voids in $(1)_\epsilon$ is $2 \sum (\epsilon^{2^n} - \epsilon^{2^{n+1}}) = 2\epsilon$, which equals to the length of $(1 - \epsilon, 1 + \epsilon)$ in R , so that the Lebesgue measure of this Cantor set is effectively zero.

Notice that our iteration process, leading to the new solution of equation (1), not only reveals a *multiplicative* structure of the nontrivial neighbourhood of 1, but also factors the differential operator in (1) into a countably infinite set of self similar operators over nonlinear scales t'_n . Accordingly, the corresponding integration measure defined on the said Cantor set, incorporating random inversions, is defined by the following replacement

$$\int_{1-\eta}^1 d \ln t_- \rightarrow \mathcal{E}_{\mathbf{R}} \int_{1-\eta}^1 d \ln t_- \equiv \sum_1^\infty \int_1^{t'_{n+}^{-1}} d \ln t'_{n+}, \tag{10}$$

where η is an infinitesimal variable, so that the new solution of equation (1) $\tau_g = C \prod \frac{1}{t'_{n+}}$, $C = \prod t'_{n+}(0)$ [5] is retrieved by direct integration from equation (1). Notice that 1 in the integral of the r.h.s. of equation (10) corresponds actually to 1_n with the definition that $\ln 1_n = 0$ for each n . More generally, the extended integral for a function f in \mathbf{R} is defined by

$$\mathcal{E}_{\mathbf{R}} \int_{\mathbf{a}}^{\mathbf{t}} f(\mathbf{t})d\mathbf{t} = \int_a^t f(t)dt + \epsilon \sum \int_1^{t'_{n+}^{-1}} [f(t'_n) + \tau_g t \frac{df}{dt}] d \ln t'_n, \tag{11}$$

where we neglect higher order infinitesimals, and use rescaling invariance $t \rightarrow t/a$ of the logarithmic differential. We also make use of $\tau_g = t \frac{d\tau_g}{dt}$, $d\tau_g = \tau_g \sum d \ln t'_n$ and $\mathbf{t} = t + \epsilon \tau_g$. Notice that the second term in the bracketed integral arises from the first order Taylor's expansion of $f(\mathbf{t})$ and the extended integral can be considered as the anti-derivative of the ODE

$$\frac{dF}{d\mathbf{t}} = f(\mathbf{t}) \quad (12)$$

in \mathbf{R} (c.f. equation (4)). As an example, the modulated exponential (7) is recovered when equation (6) is integrated following the generalized integral (11) with $f(t) = 1$.

Remark 4. Clearly, the infinite sum in (11) is uniformly convergent in every closed interval near $t = 1$ for continuous df/dt . Let us also point out that the infinite set of rescaled nonlinear variables t'_n are intrinsically randomised, as indicated in relation to equation (4), implicitly by $f(t)$ itself. Consequently, the extension of integral as defined in equation (12) should be interpreted as a generalized integral over a set with small scale (infinitesimal) random elements. We make a more rigorous treatment of this generalized integral elsewhere.

Proceeding at this heuristic level (i.e., forgetting the randomness so that all the variables t and t'_n are well defined functions of the real variable η , $0 < \eta \ll 1$) we now reexamine the Picard's Theorem in this new light.

Let us consider equation (4), viz.,

$$\frac{d \ln \tau}{d \ln t} = f(t, \tau), \quad \tau(1) = \tau_0, \quad (13)$$

where f is assumed to be a C^∞ function, for simplicity. The Picard's Theorem then guarantees a unique solution in the neighbourhood of $t = 1$. It also tells that the corresponding solution is C^∞ . The proof requires one to convert the IVP (13) to the equivalent integral equation

$$\ln \tau = \ln \tau_0 + \int_1^t f(t, \tau) d \ln t, \quad (14)$$

and then to construct a sequence of approximations $\tau_n(t)$ satisfying

$$\ln \tau_n = \ln \tau_0 + \int_1^t f(t, \tau_{n-1}) d \ln t, \quad (15)$$

converging uniformly in a closed interval within the said neighbourhood to the required solution. One verifies that the proof of the Picard's Theorem applies

also to the generalized integral (11) without major modifications. Consequently, we state the modified Picard's Theorem in the SL(2,R) analysis as follows.

Theorem. *Let $f(\mathbf{t}, \tau)$ be C^{2^n-1} in \mathbf{R} . Then the ODE*

$$\frac{d\tau}{d\mathbf{t}} = f(\mathbf{t}, \tau), \tau(1) = \tau_0 \quad (16)$$

has a unique C^{2^n-1} solution in a suitable neighbourhood of $\mathbf{t} = 1$.

Remark 5. The function f in the above theorem inherits the differentiable structure of \mathbf{t} . If instead $f \in C^m$, $m < 2^n - 1$ then the solution would obviously be only C^m .

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Appendix

The new solution τ_N reduces to the standard solution τ_s if the residual rescalings are performed only up to a finite number of iterations. Let the rescaling be terminated at the n -th iteration, so that $t'_{(n+1)+} = 1 + \eta_{n+1}$, $\eta_{n+1} = \alpha_n^2 \eta_n'^2$ [5]. Subsequent iterations (without rescalings) would then lead to a factor $1 - \eta_{(n+1)}^2$ in τ_N , viz.,

$$\tau_{N-} = C \frac{1}{t_+} \frac{1}{t'_{2+}} \cdots \frac{1}{t'_{n+}} (1 - \eta_{(n+1)}^2).$$

Consequently, the uncertainty (or twist, to put it in a pictorial way) introduced in the solution through each nontrivial rescaling and subsequent inversion would *unwind* gradually, so to speak, leading to the ordinary (nonrandom) τ_s (multiplied by n irrelevant scale factors). Things, however, change drastically if rescalings are performed an infinite number of times, instead. Indeed, the statement n tends to infinity means, of course, that n is becoming larger and larger, signifying an *unending* process. As a result, the unwinding can never be initiated in the iteration process for an infinitely large n .