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ON PRIME $\Gamma M-$SUBMODULES OF $\Gamma M-$MODULES

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Abstract: Let G be a ΓM-module. A ΓM-submodule N of G is said to be prime if for any ideal I of M and for any ΓM-submodule V of G, $V\Gamma I \subseteq N$ implies $V \subseteq N$ or $I \subseteq (N : G)_M$. The purpose of this paper is to introduce interesting properties of prime ΓM-submodules of ΓM-modules.

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Let M and Γ be two Abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$, the conditions (1) $x\alpha y \in M$; (2) $(x + y)\,\alpha z = x\alpha z + y\alpha z$, $x\,(\alpha + \beta)\,z = x\alpha z + x\beta z$, $x\alpha\,(y + z) = x\alpha y + x\alpha z$; (3) $(x\alpha y)\,\beta z = x\alpha\,(y\beta z)$ are satisfied, then we call M a Γ-ring.

A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right ideal and a left ideal, then we say that I is an ideal of M. For each a of a Γ-ring M, the smallest ideal containing a is called the principal ideal generated by a and is denoted by (a).

If A is a right ideal of M, B a left ideal of M, and S is any nonempty subset of M, then the set

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$$S\Gamma A=\left\{\sum_{i=1}^{n}s_i\alpha_i a_i:\ s_i\in S,\alpha_i\in\Gamma,a_i\in A,n\text{ any positive integer}\right\}$$

is a right ideal of M, $B\Gamma S$ is a left ideal of M, and $B\Gamma A$ is an ideal of M.

Let G be an additive group. If for all $g,g^*\in G$, $\gamma,\eta\in\Gamma$, $x,y\in M$ it holds that : (1) $g\gamma x\in G$; (2) $g\gamma(x\eta y)=(g\gamma x)\eta y$; (3) $(g+g^*)\gamma x=g\gamma x+g^*\gamma x$; (4) $g\gamma(x+y)=g\gamma x+g\gamma y$; then G is called a ΓM-module.

A subset N of G which is itself a ΓM-module, is called a ΓM-submodule of G. If $A\subseteq G$, $B\subseteq M$, we define $A\Gamma B=\{a\gamma b:a\in A,\gamma\in\Gamma,\,b\in B\}$. If V is a ΓM-submodule of G, I is a right ideal of M, then the set

$$V\Gamma I=\left\{\sum_{i=1}^{n}v_i\alpha_i a_i:v_i\in V,\alpha_i\in\Gamma,a_i\in I,\ n\text{ any positive integer}\right\}$$

is a ΓM-submodule of G. For each $g\in G$, the smallest ΓM-submodule containing g is called cyclic ΓM-submodule of G generated by g and is denoted by (g).

Let N be a submodule of G. Then we define $(N:G)_M=\{x\in M:G\Gamma x\subseteq N\}$. Note that $(N:G)_M$ is an ideal of M. The ΓM-submodule N of G is called prime if for any ideal I of M and for any ΓM-submodule V of G, $V\Gamma I\subseteq N$ implies $V\subseteq N$ or $I\subseteq(N:G)_M$. G is called prime ΓM-module if (0) is prime ΓM-submodule of G. For the properties of prime ΓM-modules, see [2].

The purpose of this paper is to introduce interesting and useful properties of prime ΓM-submodules of ΓM-modules.

Definition 1. Let G be a ΓM-module. A ΓM-submodule N of G is said to be prime if for any ideal I of M and for any ΓM-submodule V of G, $V\Gamma I\subseteq N$ implies $V\subseteq N$ or $I\subseteq(N:G)_M$.

Theorem 2. *Let G be a ΓM-module. A ΓM-submodule N of G is prime if and only if $(g)\,\Gamma\,(m)\subseteq N$ implies $g\in N$ or $m\in(N:G)_M$.*

Proof. Let N be a prime ΓM-submodule of G. Let $(g)\,\Gamma\,(m)\subseteq N$, where $g\in G$ and $m\in M$. Since N is a prime ΓM-submodule of G, $(g)\subseteq N$ or $(m)\subseteq(N:G)_M$. Therefore, $g\in N$ or $m\in(N:G)_M$. Conversely, suppose $(g)\,\Gamma\,(m)\subseteq N$ implies $g\in N$ or $m\in(N:G)_M$. Suppose that $V\Gamma I\subseteq N$ and $V\nsubseteq N$, where V is a ΓM-submodule of G and I is an ideal of M. Then there exists $v\in V$ such that $v\notin N$, and for any $a\in I$ we have $(v)\,\Gamma\,(a)\subseteq V\Gamma I\subseteq N$, whence $a\in(N:G)_M$. Thus $I\subseteq(N:G)_M$ and N is prime. □

Recall that an ideal P of Γ-ring M is called prime if for any ideals I and J of M, $I\Gamma J\subseteq P$ implies $I\subseteq P$ or $J\subseteq P$.

Lemma 3. *Let G be a ΓM-module. Let N be a prime ΓM-submodule of G. Then $(N:G)_M$ is a prime ideal of M.*

Proof. Let I and J be ideals of M such that $I\Gamma J \subseteq (N:G)_M$. Then $G\Gamma I\Gamma J \subseteq N$. Since N is a prime ΓM-submodule of G, $G\Gamma I \subseteq N$ or $J \subseteq (N:G)_M$. Therefore, $I \subseteq (N:G)_M$, or $J \subseteq (N:G)_M$. □

Definition 4. A ΓM-module G is called multiplication module if, for any ΓM-submodule N, there exists an ideal I of M such that $N = G\Gamma I$.

Theorem 5. *Let G be a multiplication ΓM-module. Then a ΓM-submodule N of G is prime if and only if $(N:G)_M$ is a prime ideal of M.*

Proof. Let N be a prime ΓM-submodule of G. Then $(N:G)_M$ is a prime ideal of M by Lemma 3. Conversely, suppose that $(N:G)_M$ is a prime ideal of M. Let $V \nsubseteq N$ and $I \nsubseteq (N:G)_M$ but $V\Gamma I \subseteq N$, where V is a $\Gamma M-$submodule of G and I is an ideal of M. Since G is a multiplication module, $V = G\Gamma J$, where J is an ideal of M. Then $V\Gamma I = G\Gamma J\Gamma I$ and so $J\Gamma I \subseteq (N:G)_M$. Since $(N:G)_M$ is a prime ideal of M and $I \nsubseteq (N:G)_M$, $J \subseteq (N:G)_M$. Therefore, $V = G\Gamma J \subseteq N$. This is a contradiction. □

If S is any subset of the Γ-ring M, we call S an m-system if $S = \emptyset$ or if a and b in S implies that $(a)\,\Gamma\,(b) \cap S \neq \emptyset$ (see [1], for more details).

Definition 6. Let S be an m-system of a Γ-ring M and G be a ΓM-module. A subset S^* of G is said to be S-system if $S^* = \emptyset$ or $(g)\,\Gamma\,(m) \cap S^* \neq \emptyset$ for every $g \in S^*$ and $m \in S$.

Theorem 7. *Let $\emptyset \neq S$ be an m-system of Γ-ring M and $\emptyset \neq S^*$ be an S-system subset of G. Let N be a ΓM-submodule of G which is maximal in $G - S^*$. If the ideal $(N:G)_M$ is maximal in $M - S$, then N is a prime submodule of G.*

Proof. Let $g \notin N$ and $m \notin (N:G)_M$ but $(g)\,\Gamma\,(m) \subseteq N$. Then $(N + (g)) \cap S^* \neq \emptyset$ and $((N:G)_M + (m)) \cap S \neq \emptyset$. There exist $t^* \in (N + (g)) \cap S^*$ and $l \in ((N:G)_M + (m)) \cap S$. Therefore,

$$\begin{aligned}(t^*)\,\Gamma\,(l) &\subseteq (N + (g))\,\Gamma\,((N:G)_M + (m)) \\ &= N\Gamma\,(N:G)_M + N\Gamma\,(m) + (g)\,\Gamma\,(N:G)_M + (g)\,\Gamma\,(m) \subseteq N.\end{aligned}$$

This is a contradiction. N is a prime submodule of G. □

Corollary 8. *Let p be a prime ideal of a Γ-ring M, $S = M - p$, and S^* an S-system subset of G. A ΓM-submodule N of G which is maximal in M-S^* is prime if $(N:G)_M = p$.*

If A and B are ideals of a Γ-ring M, and p is a prime ideal of M such that $A \cap B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. This statement is not necessarily true if "ideals of M" is replaced by "ΓM-submodules of G".

Lemma 9. *Let G be a multiplication ΓM-module. Let N be a ΓM-submodule of G. Then $N = G\Gamma (N : G)_M$.*

Proof. Let N be a ΓM-submodule of G. Then $N = G\Gamma I$ for some ideal I of M. Therefore $I \subseteq (N : G)_M$. Then $N = G\Gamma I \subseteq G\Gamma (N : G)_M \subseteq N$. Consequently, $N = G\Gamma (N : G)_M$. □

Theorem 10. *Let G be a multiplication ΓM-module. Let N be a prime ΓM-submodule of G such that $N_1 \cap N_2 \subseteq N$, where N_1 and N_2 are ΓM-submodules of G. Then $N_1 \subseteq N$ or $N_2 \subseteq N$.*

Proof. Let N be a prime ΓM-submodule of G and $N_1 \cap N_2 \subseteq N$, where N_1 and N_2 are ΓM-submodules of G. Then $(N_1 \cap N_2 : G)_M = (N_1 : G)_M \cap (N_2 : G)_M \subseteq (N : G)_M$. Since $(N : G)_M$ is a prime ideal of M, $(N_1 : G)_M \subseteq (N : G)_M$ or $(N_2 : G) \subseteq (N : G)_M$. Thus, $G\Gamma (N_1 : G)_M \subseteq G\Gamma (N : G)_M$ or $G\Gamma (N_2 : G)_M \subseteq G\Gamma (N : G)_M$. Since G is a multiplication ΓM-module, $N_1 \subseteq N$ or $N_2 \subseteq N$ by Lemma 9. □

Let I be an ideal of M. The radical of I, denoted by $\mathrm{rad}(I)$, is defined to be intersection of all prime ideals containing I. Let B be a ΓM-submodule of G. The radical of B, denoted by $\mathrm{rad}^* (B)$, is defined to be the intersection of all prime ΓM-submodules containing B and $\mathrm{rad}^* (B) = G$ if B is not contained in any prime ΓM-submodule of G.

A prime ΓM-submodule N of G is said to be p-prime if $(N : G)_M = p$.

Theorem 11. *Let G be a ΓM-modules.Let N and L be ΓM-submodules of G. Then*

(1) $N \subseteq \mathrm{rad}^* (N)$.

(2) $\mathrm{rad}^* (N \cap L) \subseteq \mathrm{rad}^* (N) \cap \mathrm{rad}^* (L)$.

(3) $\mathrm{rad}^* (G\Gamma I) = \mathrm{rad}^* (G\Gamma \mathrm{rad}\ (I))$ *for some ideal I of M.*

Proof. (1) and (2) are trivial. (3) is trivially true if $\mathrm{rad}^* (G\Gamma I) = G$. If there exists a p-prime ΓM-submodule N which contains $G\Gamma I$, then $I \subseteq (G\Gamma I : G)_M \subseteq (N : G)_M = p$. Hence, $\mathrm{rad}\ (I) \subseteq p$ so that $G\Gamma \mathrm{rad}\ (I) \subseteq G\Gamma p \subseteq N$. It follows that $\mathrm{rad}^* (G\Gamma \mathrm{rad}\ (I)) \subseteq N$ for every prime ΓM-submodule N containing $G\Gamma I$. Then $\mathrm{rad}^* (G\Gamma \mathrm{rad}\ (I)) \subseteq \mathrm{rad}^* (G\Gamma I)$. It is clear that $\mathrm{rad}^* (G\Gamma I) \subseteq \mathrm{rad}^* (G\Gamma \mathrm{rad}\ (I))$. Consequently, $\mathrm{rad}^* (G\Gamma I) = \mathrm{rad}^* (G\Gamma \mathrm{rad}\ (I))$. □

Theorem 12. *Let G be a ΓM-module. Let N and L be ΓM-submodules of G such that whenever $N \cap L \subseteq P$ we have $N \subseteq P$ or $L \subseteq P$ for any prime*

ΓM-submodule P of G. Then $\operatorname{rad}^*(N\cap L)=\operatorname{rad}^*(N)\cap\operatorname{rad}^*(L)$.

Proof. If $\operatorname{rad}^*(N\cap L)=G$, then clearly $\operatorname{rad}^*(N)=\operatorname{rad}^*(L)=G$ and so $\operatorname{rad}^*(N\cap L)=\operatorname{rad}^*(N)=\operatorname{rad}^*(L)=G$. If $\operatorname{rad}^*(N\cap L)\neq G$, then there exists a prime ΓM-submodule P such that $N\cap L\subseteq P$. By hypothesis, $N\subseteq P$ or $L\subseteq P$ so that $\operatorname{rad}^*(N)\subseteq P$ or $\operatorname{rad}^*(L)\subseteq P$. Since this is true for all prime ΓM-submodules P containig $N\cap L$, $\operatorname{rad}^*(N)\cap\operatorname{rad}^*(L)\subseteq\operatorname{rad}^*(N\cap L)$ and therefore $\operatorname{rad}^*(N\cap L)=\operatorname{rad}^*(N)\cap\operatorname{rad}^*(L)$. □

Corollary 13. *Let G be a multiplication ΓM-module. Let N and L be ΓM-submodules of G. Then* $\operatorname{rad}^*(N\cap L)=\operatorname{rad}^*(N)\cap\operatorname{rad}^*(L)$.

Definition 14. Let G be a ΓM-module. A ΓM-submodule N of G is said to be primary if for any ΓM-submodule V of G and for any ideal I of M, $V\Gamma I\subseteq N$ and $V\nsubseteq N$ implies $I\subseteq\operatorname{rad}\left((N:G)_M\right)$.

Theorem 15. *Let G be a ΓM-module. A ΓM-submodule N of G is primary if and only if $(g)\Gamma(m)\subseteq N$ and $g\notin N$, where $g\in G$ and $m\in M$ implies $m\in\operatorname{rad}\left((N:G)_M\right)$.*

Proof. The "only if" part follows trivially from the definition of a primary ΓM-submodule. So suppose that $(g)\Gamma(m)\subseteq N$ and $g\notin N$ implies $m\in\operatorname{rad}\left((N:G)_M\right)$. Let $V\Gamma I\subseteq N$ and $V\nsubseteq N$. Then there exists $v\in V$ such that $v\notin N$ and for any $a\in I$ we have $(v)\Gamma(a)\subseteq V\Gamma I\subseteq N$, hence $a\in\operatorname{rad}\left((N:G)_M\right)$, $I\subseteq\operatorname{rad}\left((N:G)_M\right)$ and N is primary. □

Proposition 16. *Let G be a ΓM-module. If N is a prime ΓM-submodule of G, then N is a primary ΓM-submodule of G.*

Proof. Let N be a prime ΓM-submodule of G. Let $V\Gamma I\subseteq N$ and $V\nsubseteq N$, where V is a ΓM-submodule of G and I is an ideal of M. Since N is a prime ΓM-submodule of G, $I\subseteq(N:G)_M$. Therefore, $I\subseteq\operatorname{rad}\left((N:G)_M\right)$. □

The ideal I of M is called a radical if $\operatorname{rad}(I)=I$.

Theorem 17. *Let G be a ΓM-module. If Q is a primary ΓM-submodule of G such that $(Q:G)_M$ is a radical ideal, then Q is a prime ΓM-submodule of G.*

Proof. Suppose that Q be a primary ΓM-submodule of G such that $(Q:G)_M$ is a radical ideal. Suppose $(m)\Gamma(r)\subseteq Q$ with $m\notin Q$, where $m\in G$ and $r\in M$. Since Q is primary, $r\in\operatorname{rad}\left((Q:G)_M\right)$. Since $(Q:G)_M$ is a radical ideal of M, $(Q:G)_M=\operatorname{rad}\left((Q:G)_M\right)$. Therefore, $r\in(Q:G)_M$. Consequently, Q is a prime ΓM-submodule of G. □

Theorem 18. *Let G be a ΓM-module. Let Q_1 and Q_2 be primary ΓM-*

submodules of G. *Let* rad $((Q_1 : G)_M) =$ rad $((Q_2 : G)_M) = p$. *Then* $Q_1 \cap Q_2$ *is primary.*

Proof. Let $V\Gamma I \subseteq Q_1 \cap Q_2$ and $V \nsubseteq Q_1 \cap Q_2$, where V is a ΓM-submodule of G and I is an ideal of M. We may assume that $V \nsubseteq Q_1$,and $V\Gamma I \subseteq Q_1$, then implies $I \subseteq$ rad $((Q_1 : G)_M) = p$. Since

$$\begin{aligned} \text{rad } ((Q_1 \cap Q_2 : G)_M) &= \text{rad } ((Q_1 : G)_M \cap (Q_2 : G)_M) \\ &= \text{rad } ((Q_1 : G)_M) \cap \text{rad } ((Q_2 : G)_M) = p, \end{aligned}$$

$I \subseteq$ rad $((Q_1 \cap Q_2 : G)_M)$. □

Definition 19. Let P be an ideal of M. P is called primary ideal of M if for any ideals I and J of M, $J\Gamma I \subseteq P$ and $J \nsubseteq P$ implies $I \subseteq$ rad (P).

Lemma 20. *Let* G *be a* ΓM*-module. If* N *is a primary* ΓM*-submodule of* G, *the* $(N : G)_M$ *is a primary ideal of* M.

Proof. Let $I\Gamma J \subseteq (N : G)_M$ and $I \nsubseteq (N : G)_M$, where I and J are ideals of M. Then $G\Gamma I\Gamma J \subseteq N$ and $G\Gamma I \nsubseteq N$. Since N is a primary ΓM-submodule of G, $J \subseteq$ rad $((N : G)_M)$. □

Theorem 21. *Let* G *be a* ΓM*-module. If* Q *is a primary* ΓM*-submodule of* G, *then* rad $((Q : G)_M) = p$ *is prime ideal of* M.

Proof. Let Q be a primary ΓM-submodule of G. Then $(Q : G)\, M$ is a primary ideal of M by Lemma 20. Thus rad $((Q : G)_M) = p$ is a prime ideal of M (see [1], Theorem 12). □

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