International Journal of Pure and Applied Mathematics
Volume 19 No. 1 2005, 13-15

RING-THEORETIC INTERPOLATION FOR POLYNOMIALS

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Abstract: Here we study the interpolation of multi-homogeneous polynomial defined over a subring of $\mathbb{C}$.

AMS Subject Classification: 14N05, 13F20
Key Words: interpolation for polynomials, interpolation over a commutative ring

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Fix integers $s > 0$, $n_i > 0$, $1 \le i \le s$, $d_i > 0$, $1 \le i \le s$, a subring R of the complex mumber field $\mathbb{C}$ containing 1 and subsets $\Gamma_{i,j} \subseteq R$, $1 \le i \le s$, $1 \le j \le n_i$, such that $\sharp(\Gamma_{i,j}) \ge d_i + 1$ for all i, j. Use the coordinates $x_{i,j}$, $1 \le i \le s$, $1 \le j \le s$, on the affine space $\mathbb{A}^{n_1+\cdots+n_s}$ over $\mathbb{C}$. Set $\mathbb{A}_R^{n_1+\cdots+n_s} := \{P \in: \mathbb{A}^{n_1+\cdots+n_s} : x_{i,j}(P) \in R \text{ for all } i,j\}$ and $\Gamma := \{P \in: \mathbb{A}^{n_1+\cdots+n_s} : x_{i,j}(P) \in \Gamma_{i,j} \text{ for all } i,j\} = \prod_{i=1}^{s} \prod_{j=1}^{n_s} \Gamma_{i,j}$. Let $R\langle d_1, \dots, d_s\rangle$ denote the R-module of all polynomials in the variables $x_{i,j}$ with coefficients in R and such that their total degree with respect to each set of variables $x_{i,j}$, $1 \le j \le n_i$, is at most d_i. Hence $R\langle d_1, \dots, d_s\rangle$ is a free R-module with rank $\prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$. Our main interest is when R is the ring of integers of a number field. We believe that our set-up (and in particular the stress on the sets $\Gamma_{i,j}$) is quite useful to obtain by reduction modulo p explicit solutions for interpolation problems over finite

Received: November 5, 2004

fields. For the case of multi-homogeneous polynomials over $\mathbb{R}$, their application and the terminology used in applied mathematics, see [2] and references therein.

Remark 1. Since $\sharp(\Gamma_{i,j}) \geq d_i$ for all i, j the restriction map ρ_Γ defined by $f \mapsto f|\Gamma$ from $R\langle d_1, \dots, d_s\rangle$ into the interpolation space R^Γ is injective. If this R-linear map ρ_Γ is surjective, then $n_i = 1$ for all i and $\sharp(\Gamma_{i,j}) = d_i + 1$ for all i, j. The converse is true when R is a field, but not in general as the following example shows. Take $R = \mathbb{Z}$, $s = 1$, $n_1 = 1$, $d_1 = 1$ and $\Gamma_{1,1} = \{0, 2\}$: every polynomial with integer coefficients vanishing at 0 has even value at 2. This is not due to the bad choice of the set $\{0, 2\}$ (in general), but to a fundamental problem when R is not a field. Indeed, it is quite common that we are forced to take denominators. Take for instance $s = n = 1$, arbitrary d_1 and $R = \mathbb{Z}$. There is no basis of the $\mathbb{Z}$-module of all integer valued polynomial functions in one-variable of degree at most $d \geq 2$ over $\mathbb{Q}$ given by polynomials with integer coefficients: one is forced to use integer valued polynomial functions induced by suitable binomial coefficients. For any integer $\gamma > 0$ we will write R_γ for the ring of franctions of R obtained inverting γ, i.e. obtained inverting every prime dividing γ. Call $\rho_{\Gamma,\gamma}$ the restriction map from $R_\gamma\langle d_1, \dots, d_s\rangle$ into the interpolation space R_γ^Γ

Proposition 1. *Fix integers s, $n_i > 0$, $d_i > 0$, $1 \leq i \leq s$, and the ring $R \subseteq \mathbb{C}$. Let γ be the products of all primes $p \leq \prod_{i=1}^{s} \prod_{j=1}^{n_i} \binom{n_i+d_i}{n_i}$. Then there exist $\Gamma_{i,j} \subset R$ such that $\Gamma_{i,j} = d_i + 1$ and for every integer $t \leq \prod_{i=1}^{s} \prod_{j=1}^{n_i} \binom{n_i+d_i}{n_i}$ there exists $S \subset \Gamma$ such that $\sharp(S) = t$ and the restriction map $\rho_S : R\langle d_1, \dots, d_s\rangle \to H^0(S, \mathcal{O}_S(d_1, \dots, d_s))$ is injective. Now assume $R = \mathbb{Z}$ and set $\Gamma_{i,j} := \{a \in \mathbb{Z} : 0 \leq a \leq d_i + 1\}$. Then there is S as above such that $\rho_{S,\gamma}$ is injective and with free cokernel as $\mathbb{Z}_\gamma$-module.*

Proof. Taking a suitable $A_i \subset \prod_{j=1}^{n_i} \Gamma_{i,j}$ we easily reduce to the case $s = 1$. Hence we will assume $s = 1$. For the first assertion, just use the first part of Remark 1. For the second part use the last sentence of Remark 1. $\square$

Let $Z \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ be a zero-dimensional scheme. For every s-ple of integers $(t_1, \dots, t_s)$ let

$$\rho_{Z;n_1,\dots,n_s;t_1,\dots,t_s} : H^0(\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}, \mathcal{O}_{\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}}(t_1, \dots, t_s)) \to H^0(Z, \mathcal{O}_Z(t_1, \dots, t_s))$$

be the restriction map. We will say that Z is a *dot* if it is connected and $\text{length}(Z) = 2$. Now see $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ as a compactification of $\mathbb{A}^{n_1+\cdots+n_s}$. A dot will be called an *R-dot* (for this fixed choice of the compactification) if $Z_{red} \in R^{n_1+\cdots+n_s}$ and the line $\langle Z\rangle$ of $R^{n_1+\cdots+n_s}$ spanned by Z is defined over R.

An R-dot will be called a *vertical R-dot* if the following very strong condition is satified; let $x_{i,j}(P) \in R$ be the coordinates of the point $P := Z_{red}$; then there is an integer i_0 such that every $Q \in \langle Z \rangle$ is contained in the product of $\mathbb{A}^{n_{i_0}}$ has coordinates $x_{i,j}(Q) = x_{i,j}(P)$ for all $i \neq i_0$. Since R is infinite, the proof of [1], Theorem 1.1, gives easily the first assertion of the following result. The second assertion easily follows from the omitted proof. However, we do not know if the bounds $\sharp(\Gamma_{i,j}) \geq t_1 + \cdots + t_s + 1$ for all i, j is sharp; we conjecture that it is sufficient to assume $\sharp(\Gamma_{i,j}) \geq 1 + \max\{t_1, \dots, t_s\}$ for all i, j.

Theorem 1. *Fix integers $s > 0$, $n_i \geq 2$, $t_i > 0$, $1 \leq i \leq s$, and z such that $2z \leq \prod_{i=1}^{s} \prod j = 1^{n_i} \binom{n_i+t_i}{n_i}$. Then there is a union $Z \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ of z vertical R-dots such that $h^1(\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}, \mathcal{I}_Z(t_1, \dots, t_s)) = 0$. Furthermore, for every $\Gamma_{i,j} \subset R$ such that $\sharp(\Gamma_{i,j}) \geq t_1 + \cdots + t_s + 1$ for all i, j there is Z as above such that $Z_{red} \subseteq \prod_{i=1}^{s} \prod_{j=1}^{n_i} \Gamma_{i,j}$.*

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

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[2] R.A. Lorentz, *Multivariate Birkhoff Interpolation*, Lect. Notes in Math., **1516**, Springer, Berlin-Heidelberg-New York (1992).