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ON UNIFORM STATISTICAL CONVERGENCE

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Abstract: In this study, we introduce a notion of uniform statistical convergence of sequences of real-valued functions. Furthermore we introduce the concept of a uniform statistical Cauchy sequence for functional sequences and prove that it is equivalent to uniform statistical convergence of sequence of real-valued functions.

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1. Introduction and Background

The notion of statistical convergence was introduced by Fast [2] and also independently by Buck [1] and Schoenberg [6] for real and complex number sequences.

A subset A of the ordered set $\mathbf{N}$ of natural numbers is said to have density $\delta(A)$, if $\lim \frac{|A(n)|}{n} = \delta(A)$, where $A(n) = \{k < n : k \in A\}$ and $|A|$ denotes the cardinality of set $A \subset \mathbf{N}$ [5]. Clearly, finite sets have zero density and $\delta(A') = 1 - \delta(A)$, whenever either side exists and $A' = \mathbf{N} - A$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k, that is a.a. k.

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The number sequence x is statistically convergent to the number L provided that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0,$$

i.e.,

$$|x_k - L| < \varepsilon\,, \quad \text{a.a. } k.$$

In this case, we write $\mathrm{st}-\lim x_k = L$.

The sequence x is a statistically Cauchy sequence provided that for every $\varepsilon > 0$ there exists a number $N(= N(\varepsilon))$ such that

$$|x_k - x_N| < \varepsilon\,, \quad \text{a.a. } k.$$

Fridy [3] obtained an equivalent criterion for statistically convergent real sequences, similar to the Cauchy criterion of convergence.

We deal with sequences $\{f_k\}$ whose terms are real-valued functions having a common domain on the real line $\mathbf{R}$. For each x in the domain, we can form another sequence $(f_k(x))$ whose terms are the corresponding function values. Let S denote the set of x for which this second sequence converges. The function f defined by the equation

$$f(x) = \lim_{k\to\infty} f_k(x)\,, \text{ uniformly on } S,$$

is called the limit function of the sequence $\{f_k\}$, and we say that $\{f_k\}$ converges uniformly to f on the set S. That means that for all points x in S and for each $\varepsilon > 0$, there exists a K (depending only on ε) such that $k > K$ implies

$$|f_k(x) - f(x)| < \varepsilon.$$

Recently, Gökhan and Güngör [4] defined pointwise statistical convergence by saying that $\mathrm{st}-\lim f_k(x) = f(x)$ or $f_k \xrightarrow{st.} f$ on S if and only if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}\right| = 0,$$

i.e., for every $x \in S$,

$$|f_k(x) - f(x)| < \varepsilon\,, \quad \text{a.a. } k.$$

2. Uniform Statistical Convergence

In this section we first introduce uniform statistical convergence. Furthermore we give the relations between uniform statistical convergence and pointwise statistical convergence, uniform convergence.

Definition 2.1. A sequence of functions $\{f_k\}$ is said to be uniformly statistically convergent to f on a set S if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |f_k(x) - f(x)| \ge \varepsilon \text{ for all } x \in S\}\right| = 0,$$

i.e., for all $x \in S$,

$$|f_k(x) - f(x)| < \varepsilon\,, \quad \text{a.a. } k. \tag{1}$$

In this case, we write

$$\text{st} - \lim f_k(x) = f(x) \text{ uniformly on } S$$

or

$$f_k \xrightarrow{st.} f$$

uniformly on S.

Now, let us give the following theorem which characterizes the sequence of $(f_k(x))$ – uniform statistical convergent or not.

Theorem 2.2. *Let f and f_k, $k = 1, 2, ...,$ be continuous functions on $I = [a, b] \subset \mathbf{R}$. Then* $\text{st}-\lim f_k(x) = f(x)$ *uniformly on I if and only if* $\text{st}-\lim c_k = 0$, *where* $c_k = \max\limits_{x\in I} |f_k(x) - f(x)|$.

Proof. Suppose that $\text{st}-\lim f_k(x) = f(x)$ uniformly on I. Since $|f_k(x) - f(x)|$ is continuous on I for each $k \in \mathbf{N}$, it has absolute maximum value at some point $x_k \in I$, i.e. there exist $x_1, x_2, ... \in I$ such that $c_1 = |f_1(x_1) - f(x_1)|$, $c_2 = |f_2(x_2) - f(x_2)|$, ..., etc. Thus, we may write $c_k = |f_k(x_k) - f(x_k)|$, $k = 1, 2, ...$ From the definition of uniform statistical convergence, we may write, for every $\varepsilon > 0$

$$|f_k(x_k) - f(x_k)| < \varepsilon\,, \quad \text{a.a. } k.$$

Hence, $\text{st}-\lim c_k = 0$.

The necessity is trivial. □

It is clear that if the inequality in (1) holds for all but finitely many k, then $\lim f_k(x) = f(x)$ uniformly on S. It easily follows that $\lim f_k(x) = f(x)$ uniformly on S implies st-lim $f_k(x) = f(x)$ uniformly on S. But the converse is not true, as the following example shows.

Let

$$f_k(x) = \begin{cases} 3, & k = n^2,\ n = 1, 2, \dots, \\ \left(x - \frac{1}{k}\right)^2, & \text{otherwise}, \end{cases}$$

if $x \in [-1, 1]$, $k = 1, 2, \dots$. $\{f_k\}$ is uniformly statistically convergent to $f(x) = x^2$ on $[-1, 1]$ since $\text{st}-\lim c_k = 0$, where

$$c_k = \max_{x\in[-1,1]} \left|f_k(x) - x^2\right| = \begin{cases} 3, & k = n^2,\ n = 1, 2, \dots, \\ \frac{2}{k} + \frac{1}{k^2}, & \text{otherwise}. \end{cases}$$

But $(f_k(x))$ is not uniformly convergent on $[-1, 1]$ since $\lim_{k\to\infty} c_k$ does not exist.

Furthermore, it is clear that uniform statistical convergence implies pointwise statistical convergence with the same limit f on the set S. Note that the converse of this theorem is not true. For this, let us consider the following example.

Let

$$f_k(x) = \begin{cases} 1, & k = n^2,\ n = 1, 2, \dots, \\ \frac{k^2 x}{1+k^3x^2}, & \text{otherwise}, \end{cases}$$

if $x \in [0, 1]$, $k = 1, 2, \dots$. Then the sequence $\{f_k\}$ is pointwise statistically convergent to $f(x) = 0$ on $[0, 1]$. But $\{f_k\}$ is not uniformly statistically convergent by Theorem 2.2. Because

$$c_k = \max_{x\in[0,1]} |f_k(x) - 0| = \begin{cases} 1, & k = n^2,\ n = 1, 2, \dots, \\ \frac{\sqrt{k}}{2}, & \text{otherwise} \end{cases}$$

and $\text{st}-\lim c_k$ does not exist.

Now, we can give the relations between well-known convergence models and our studied models as the following result.

Corollary 2.3. *i)* $\lim f_k(x) = f(x)$ *uniformly on* $S \Rightarrow \lim f_k(x) = f(x)$ *on* $S \Rightarrow \text{st} - \lim f_k(x) = f(x)$ *on* S.

ii) $\lim f_k(x) = f(x)$ *uniformly on* $S \Rightarrow \text{st} - \lim f_k(x) = f(x)$ *uniformly on* $S \Rightarrow \text{st} - \lim f_k(x) = f(x)$ *on* S.

Theorem 2.4. *Let* $\{f_k\}$ *and* $\{g_k\}$ *be two sequences of functions defined on a set* S. *If* $\text{st} - \lim f_k(x) = f(x)$ *and* $\text{st}-\lim g_k(x) = g(x)$ *uniformly on* S, *then* $\text{st} - \lim(\alpha f_k(x) + \beta g_k(x)) = \alpha f(x) + \beta g(x)$ *uniformly on* S, *where* $\alpha, \beta \in \mathbf{R}$.

Proof. The proof is similar to that of Theorem 2.1 in [4]. Therefore we omit it. □

3. The Cauchy Condition for Uniform Statistical Convergence

In most convergence theories it is desirable to have a criterion that can be used to verify convergence without using the value of the limit. For this purpose we introduce the statistical analog of the Cauchy convergence criterion.

Definition 3.1. Let $\{f_k\}$ be a sequence of functions on a set S. The sequence $\{f_k\}$ is a uniformly statistically Cauchy sequence provided that for every $\varepsilon > 0$ there exists a number $N(= N(\varepsilon))$ such that

$$|f_k(x) - f_N(x)| < \varepsilon\,, \quad \text{a.a. } k. \text{ for all } x \in S$$

i.e.

$$\lim_{n\to\infty} \frac{1}{n}\,|\{k \le n : |f_k(x) - f_N(x)| \ge \varepsilon \text{ for all } x \in S\}| = 0.$$

Gökhan and Güngör [4] obtained an equivalent criterion for statistical pointwise convergence of a sequence of real-valued functions, similar to the Cauchy criterion of convergence. We give a similar criterion for uniform statistical convergence as follows.

Theorem 3.2. *Let $\{f_k\}$ be a sequence of functions defined on a set S. The following statements are equivalent:*

i) $\{f_k\}$ is a uniformly statistically convergent sequence on S;

ii) $\{f_k\}$ is a uniformly statistically Cauchy sequence on S;

iii) $\{f_k\}$ is a sequence of functions for which there is a uniformly convergent sequence of function $\{g_k\}$ such that $f_k(x) = g_k(x)$ a.a. k for all $x \in S$.

Proof. To prove that (i) implies (ii) we use an adaptation of the familiar proof that a convergent sequence is a Cauchy sequence. Suppose that $\text{st}-\lim f_k(x) = f(x)$ uniformly on S and let $\varepsilon > 0$. Then $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ a.a. k for all $x \in S$, and if $\mathbf{N}$ is chosen so that $|f_N(x) - f(x)| < \frac{\varepsilon}{2}$, then we have

$$|f_k(x) - f_N(x)| \le |f_k(x) - f(x)| + |f_N(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\,, \quad \text{a.a. } k$$

for all $x \in S$. Hence $\{f_k\}$ is a uniformly statistically Cauchy sequence on S.

Next, assume (ii) is true and choose $\mathbf{N}$ so that the band $I = [f_N(x) - 1, f_N(x) + 1]$ contains $f_k(x)$ a.a. k for all $x \in S$. Also apply (ii) to choose M so that $I' = [f_M(x) - \frac{1}{2}, f_M(x) + \frac{1}{2}]$ contains $f_k(x)$ a.a. k for all $x \in S$. We assert that

$$I_1 = I \cap I' \text{contains } f_k(x)\,, \quad \text{a.a. } k \quad \text{for all } x \in S\,,$$

for

$$\left\{k \le n : f_k(x) \notin I \cap I' \text{ for all } x \in S\right\}$$
$$= \{k \le n : f_k(x) \notin I \text{ for all } x \in S\} \cup \left\{k \le n : f_k(x) \notin I' \text{ for all } x \in S\right\},$$

so,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : f_k(x) \notin I \cap I' \text{ for all } x \in S\right\}\right|$$
$$\le \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : f_k(x) \notin I \text{ for all } x \in S\}\right|$$
$$+ \lim_{n\to\infty} \frac{1}{n} \left\{k \le n : f_k(x) \notin I' \text{ for all } x \in S\right\} = 0.$$

Therefore I_1 is closed band of height less than or equal to 1 that contains $f_k(x)$ a.a. k for all $x \in S$. Now we proceed by choosing $N(2)$ so that $I'' = [f_{N(2)}(x) - \frac{1}{4}, f_{N(2)}(x) + \frac{1}{4}]$ contains $f_k(x)$ a.a. k, and by the preceding argument $I_2 = I_1 \cap I''$ contains $f_k(x)$ a.a. k for all $x \in S$, and I_2 has height less than or equal to $\frac{1}{2}$. Continuing inductively we construct a sequence $\{I_m\}_{m=1}^{\infty}$ of closed band such that for each m, $I_m \supseteq I_{m+1}$, the height of I_m is not greater than 2^{1-m}, and $f_k(x) \in I_m$ a.a. k for all $x \in S$. Thus, there exists a function $f(x)$, defined on S, that $\{f(x)\}$ is equal to $\bigcap_{m=1}^{\infty} I_m$. Using the fact that $f_k(x) \in I_m$ a.a. k for all $x \in S$, we choose an increasing positive sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{n} \left|\{k \le n : f_k(x) \notin I_m\}\right| < \frac{1}{m} \text{ if } n > T_m. \tag{2}$$

Now define a subsequence $(z_k(x))$ of $(f_k(x))$ consisting of all terms $f_k(x)$ such that $k > T_1$ and if $T_m < k \le T_{m+1}$ then $f_k(x) \notin I_m$.

Next define the sequence of function $(g_k(x))$ by

$$g_k(x) = \begin{cases} f(x), & \text{if } f_k(x) \text{ is a term of } (z_k(x)), \\ f_k(x), & \text{otherwise} \end{cases}$$

for all $x \in S$. Then $\lim_{k\to\infty} g_k(x) = f(x)$ on S; for, if $\varepsilon > \frac{1}{m} > 0$ and $k > T_m$ then either $f_k(x)$ is a term of $(z_k(x))$, which means that $g_k(x) = f(x)$, or $g_k(x) = f_k(x) \in I_m$ on S and $|g_k(x) - f(x)| \le$ height of $I_m \le 2^{1-m}$ for all $x \in S$. We also assert that $f_k(x) = g_k(x)$ a.a. k for all $x \in S$. To verify this observe that if $T_m < n < T_{m+1}$, then

$$\{k \le n : g_k(x) \ne f_k(x) \text{ for all } x \in S\}$$
$$\subseteq \{k \le n : f_k(x) \notin I_m \text{ for all } x \in S\}.$$

So, by (2)

$$\frac{1}{n}\left|\{k \leq n : g_k(x) \neq f_k(x) \text{ for all } x \in S\}\right| \\ \leq \frac{1}{n}\left|\{k \leq n : f_k(x) \notin I_m \text{ for all } x \in S\}\right| < \frac{1}{m}.$$

Hence, the limit is 0 as $n \to \infty$ and $f_k(x) = g_k(x)$ a.a. k for all $x \in S$. Therefore (ii) implies (iii).

Finally, assume that (iii) holds, say $f_k(x) = g_k(x)$ a.a. k for all $x \in S$ and $\lim_{k\to\infty} g_k(x) = f(x)$ on S. Let $\varepsilon > 0$. Then for each n, $\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon$ for all $x \in S\} \subseteq \{k \leq n : f_k(x) \neq g_k(x)$ for all $x \in S\} \cup \{k \leq n : |g_k(x) - f(x)| \geq \varepsilon$ for all $x \in S\}$; since $\lim_{k\to\infty} g_k(x) = f(x)$ on S, the latter set contains a fixed number of integers, say $l = l(\varepsilon, x)$. Therefore

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \text{ for all } x \in S\}\right| \\ \leq \lim_{n\to\infty} \frac{1}{n}\left|\{k \leq n : f_k(x) \neq g_k(x) \text{ for all } x \in S\}\right| + \lim_{n\to\infty} \frac{l}{n} = 0$$

because $f_k(x) = g_k(x)$ a.a. k for all $x \in S$. Hence $|f_k(x) - f(x)| < \varepsilon$ a.a. k for all $x \in S$, so (i) holds and the proof is complete. □

Corollary 3.3. *If $\{f_k\}$ is a sequence of functions such that*

$$\text{st} - \lim f_k(x) = f(x)\,,$$

uniformly on S, then $\{f_k\}$ has a subsequence $(f_{k(i)}(x))$ such that

$$\lim_{i\to\infty} f_{k(i)}(x) = f(x)\,,$$

uniformly on S.

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