International Journal of Pure and Applied Mathematics

Volume 19 No. 1 2005, 43-47

ON REGULAR MODULE

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Abstract: In this paper we shall consider some properties of regular modules. We show that every submodule of a divisible regular module over an integral domain is divisible and we also show that $J(R)M = 0$ for every regular R-module M. It is shown that if every simple R-module is regular then $J(R) = \cap \operatorname{ann}(M)$, where the intersection is over all regular R-module. If $J(R) \neq 0$ and M is a finitely generated regular R-module, then every prime submodule of M is maximal. In this paper, R will be a commutative ring with identity and all modules are unitary.

AMS Subject Classification: 13C13, 13C05

Key Words: regular module, divisible regular module, simple R-module, commutative ring

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Definition 1. Let M be an R-module and $x \in M$, x is called regular if there exists an R-module homomorphism $\phi : M \longrightarrow R$ such that $\phi(x)x = x$. If every element of M is regular we say that M is regular.

Definition 2. A submodule N of M is called pure if $IM \cap N = IN$ for every ideal I of R.

Definition 3. An R-submodule N of M is called prime submodule if $rm \in N$ for $r \in R$ and $m \in M$ and $m \notin N$ then $rM \subset N$.

Received: December 14, 2004

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Lemma 1. *If M is a regular R-module then every submodule of M is a pure submodule.*

Proof. If N is a submodule of M and I is an ideal of R we show that $IM \cap N = IN$. Clearly $IN \subset IM \cap N$ and if $x \in IM \cap N$ then $x = \sum_{i=1}^{n} a_i x_i$, where $a_i \in I$ and $x_i \in M$ and since x is regular there exists $\phi : M \longrightarrow R$ such that $\phi(x)x = x$, $\phi(x) = \sum_{i=1}^{n} a_i \phi(x_i)$ and $x = \phi(x)x = \sum_{i=1}^{n} a_i \phi(x_i)x$, since $x \in N$, $\phi(x_i)x \in N$ and so $x = \sum_{i=1}^{n} a_i \phi(x_i)x \in IN$, hence $IN = IM \cap N$. □

Lemma 2. *Every direct summand of an R-module M is pure.*

Proof. Suppose N is a direct summand of M, then $M = N \oplus K$ for some submodule K of M. We show that N is pure. If I is an ideal of R, $IN \subset IM \cap N$. Now if $x \in IM \cap N$ then

$$x = \sum_{i=1}^{n} r_i(a_i + b_i) = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} r_i b_i \,,$$

where $r_i \in I$, $a_i \in N$ and $b_i \in K$, so $x - \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} r_i b_i$. Since $x \in N$, $x - \sum_{i=1}^{n} r_i a_i \in N$ and $\sum_{i=1}^{n} r_i b_i \in K$ so $\sum_{i=1}^{n} r_i b_i \in N \cap K = 0$. Therefore $x = \sum_{i=1}^{r} r_i a_i$ i.e., $x \in IN$ and $IM \cap N = IN$. □

Lemma 3. *If $M = \bigoplus_{i \in I} M_i$ then M is regular if and only if each M_i is regular.*

Proof. See [4, Corollary 11]. □

Remark. The converse of Lemma 1 is not true. We show that there exists an R-module which every submodules is pure but is not a regular R-module. Consider $M = Z_2 \oplus Z_2$ as Z-module then by Lemma 2 every submodule of M is pure. But M is not regular, because there is no non-zero homomorphism from M to Z.

Here we consider some examples.

Example 1. If $M = \bigoplus_{i \in I} Z$ as a Z-module then every submodule of M is pure but M is not regular, since Z is not regular as a direct summand of M.

Lemma 4. *A ring R is von Neuman regular iff R is regular as an R-module.*

Proof. If $x \in R$, there exists $y \in R$ such that $x = xyx$ we defined $\phi : R \longrightarrow R$, by $\phi(r) = ry$ for every $r \in R$ then $\phi(x)x = x$. For the converse, if $x \in R$

so there exists ϕ such that $\phi(x)x = x$ then $x\phi(1)x = x$ we take $\phi(1) = y$ so $xyx = x$. □

Lemma 5. *If M is a torsion free R-module and R is an injective R-module then M is regular.*

Proof. If $x \in M$ we defined $\phi : Rx \longrightarrow R$ by $\phi(rx) = r$ then ϕ is well defined and $\phi(x)x = x$. As R is injective we can extend ϕ to an R-module homomorphism $\alpha : M \longrightarrow R$ such that $\alpha|_{Rx} = \phi$ so $\alpha(x)x = x$. □

Example 2. Every vector space over a field is regular.

Lemma 6. *If M is regular and divisible over an integral domain R, then every submodule of M is divisible.*

Proof. If N is a submodule of M and $0 \neq r \in R$ we show that $rN = N$. By Lemma 1 N is a pure submodule of M so $< r > M \cap N =< r > N$.

We show that $rM \cap N = rN$, if $x \in rM \cap N$ then $x = rm$ since x is regular there exists $\phi : M \longrightarrow R$ such that $\phi(x)x = x$ and so $x = \phi(x)x = r\phi(m)x$, as $x \in N$ this implies that $x \in rN$ so $rM \cap N = rN$. As $rM = M$ we see that $rN = N$, so N is divisible. □

Lemma 7. *If M is a divisible and torsion free R-module and R is an integral domain which is not a field then M is not regular.*

Proof. Suppose M is regular then by Lemma 1 every submodule of M is pure and we have $Rx\cap < r > M =< r > Rx$ for every $x \in M$ and $r \in R$. Since M is divisible $rM = M$ for every $0 \neq r \in R$ and as $x \in Rx$ and $x \in< r > M$ $(rM = M,\ x = rm)$ so $x \in< r > Rx$ i.e., $x = \sum_{i=1}^{n} r_i m_i$, where $r_i \in< r >$, $m_i \in Rx$, then $x = \sum_{i=1}^{n} rs_i t_i x = rx \sum_{i=1}^{n} s_i t_i$ so $x(1 - r\sum_{i=1}^{n} s_i t_i) = 0$ and since M is torsion free $1 - r\sum_{i=1}^{n} s_i t_i = 0$, i.e., r is invertible that which contradicts the fact that R is not a field. □

Theorem 8. *For every regular module M we have $J(R)M = 0$ ($J(R)$ is the Jacobson radical of R).*

Proof. Since M is regular every submodule of M is pure. Let $J(R)M \neq 0$ and $x \in J(R)M$ so Rx is pure submodule of M, so $Rx \cap J(R)M = JRx$. Therefore $Rx = J(R)Rx$ and by Nakayam's Lemma $Rx = 0$, so $x = 0$ and hence $J(R)M = 0$. □

Lemma 9. *If $x \in M$ is a regular element of M then $M = Rx \oplus K$ for some submodule K of M.*

Proof. By definition of regularity there exists $\phi : M \longrightarrow R$ such that $\phi(x)x = x$. Define $K = \{y \in M | \phi(y)x = 0\}$, clearly K is a submodule of M.

Note that $y = y - \phi(y)x + \phi(y)x$, and

$$\phi(y - \phi(y)x)x = \phi(y)x - \phi(y)\phi(x)x = \phi(y)x - \phi(y)x = 0\,,$$

so $y - \phi(y)x \in K$ and $\phi(y)x \in Rx$. Now we show that $Rx \cap K = 0$. If $y \in Rx$ and $y \in K$ then $y = rx$ and $\phi(y) = r\phi(x)$, we have $0 = \phi(y)x = r\phi(x)x = rx$ so $y = 0$, hence $M = Rx \oplus K$. □

Proposition 10. *Every semisimple torsion free R-module M is regular.*

Proof. If $x \in M$ then $M = Rx \oplus K$. Let $\pi : Rx \oplus K \longrightarrow Rx$ be the canonical projection and $\alpha : Rx \longrightarrow R$, defined by $\alpha(rx) = r$. We defined $\phi = \alpha \circ \pi$. Clearly $\phi(x)x = x$ so M is regular. □

Corollary 11. *Every torsion free simple R-module is regular.*

Proof. It follows straight forward from Proposition 10. □

Theorem 12. *If every simple R-module is regular then $J(R) = \cap \operatorname{ann}(M)$, where intersection is over all regular R-modules.*

Proof. By Theorem 8 $J(R)M = 0$, so $J(R) \subset \operatorname{ann}(M)$ for all regular module M, if $x \in \cap \operatorname{ann}(M)$ for all regular module M and if $x \notin J(R)$ so $x \notin P$ for some maximal ideal P of R. Let $M = \frac{R}{P}$ so M is simple and $xM = 0$. So $x(1 + P) = P$ i.e., $x \in P$ which is not the case. So $x \in J(R)$ and then $J(R) = \cap \operatorname{ann}(M)$ for all regular R-module M. □

Lemma 13. *If M is a finitely generated regular module then $J(M) = 0$.*

Proof. If $x \in J(M)$, so by Lemma 8 $M = Rx \oplus K$ if $K = 0$ then $M \subset J(M)$ that is contradiction. So $K \neq 0$, and since Rx is small so $Rx = 0$ and hence $x = 0$, then $J(M) = 0$. □

Lemma 14. *If R is a regular ring (Von Neuman) then every prime ideal of R is maximal.*

Proof. Suppose P is a prime ideal of R and M a maximal ideal of R such that $P \subset M$. We show that $P = M$. If $x \in M$ then $x = xyx$ so $x(1 - yx) = 0 \in P$, if $x \notin P$ then $1 - yx \in P$ so $1 - yx \in M$ which implies that $1 \in M$ and its contradicts the fact that M is maximal, so $P = M$. □

Remark. Lemma 14 is not true in general case for regular modules.

Proposition 15. *If M is a finitely generated regular R-module and M is divisible and $J(R) \neq 0$ then every prime submodule of M is maximal.*

Proof. If N is a prime submodule of M since M is $f.g$, $N \subset K$ for some maximal submodule K of M. We show that $N = K$. If $x \in K - N$ then by regularity there exists $\phi : M \longrightarrow R$ such that $\phi(x)x = x$, so $(\phi(x) - 1)x = 0 \in N$ and since N is prime, so $(\phi(x) - 1)M \subset N$. If $\phi(x) = 1$ then ϕ is an epimorphism and $\frac{M}{\ker\phi} \simeq R$. By Lemma 8 $J(R)M = 0$ so $J(R)R = 0$ i.e., $J(R) = 0$ which is not the case, therefore $\phi(x) \neq 1$, so $\phi(x) - 1 \neq 0$. By divisibility $(\phi(x)-1)M = M$ and $M \subset N \subset K$, but this contradicts the maximality of K, so $K = N$. □

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