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HILBERT SEMIGROUPS

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Abstract: The notion of Hilbert semigroups is introduced, and related properties are investigated. The concept of left (resp. right) deductive systems (LDS (resp. RDS) for short) of a Hilbert semigroup is also introduced. we describe the LDS generated by a nonempty subset in a Hilbert semigroup as a simple form.

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1. Introduction

Hilbert algebras correspond to the algebraic counterpart of the implicative fragment of intuitionistic propositional logic. Important subclasses of Hilbert algebras are the so called Tarski algebras, linear Hilbert algebras, and Hilbert algebras which are join-semilattices. In this paper, by combining Hilbert algebras and semigroups, we introduce the notion of Hilbert semigroups. We define left (resp. right) deductive systems (LDS (resp. RDS) for short) of a Hilbert

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semigroup, and describe the LDS generated by a nonempty subset in a Hilbert semigroup as a simple form.

2. Preliminaries

An algebra $(A; \to, 1)$ of type (2,0) is called a *Hilbert algebra* if it satisfies:

(H1) $(\forall a, b \in A)$ $(a \to (b \to a) = 1)$.

(H2) $(\forall a, b, c \in A)$ $((a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1)$.

(H3) $(\forall a, b \in A)$ $(a \to b = b \to a = 1 \Rightarrow a = b)$.

If $(A; \to, 1)$ is a Hilbert algebra and we define a binary relation $\leq$ in $(A; \to, 1)$ by $a \leq b$ if and only if $a \to b = 1$, then $\leq$ is a partial order in $(A; \to, 1)$.

In a Hilbert algebra $(A; \to, 1)$, we have the following assertions:

(a1) $x \leq y \to x$.

(a2) $x \to 1 = 1$, $1 \to x = x$.

(a3) $x \to (y \to z) = (x \to y) \to (x \to z)$.

(a4) $x \leq (x \to y) \to y$.

(a5) $x \to (y \to z) = y \to (x \to z)$.

(a6) $x \to y \leq (y \to z) \to (x \to z)$.

(a7) $x \leq y \Rightarrow z \to x \leq z \to y$, $y \to z \leq x \to z$.

3. Hilbert Semigroups

Definition 3.1. An algebraic system $\mathbb{A} = (A; \odot, \to, 1)$ is called a *Hilbert semigroup* if it satisfies:

(i) $(A; \odot)$ is a semigroup.

(ii) $(A; \to, 1)$ is a Hilbert algebra.

(iii) The operation $\odot$ is distributive (on both sides) over the operation $\to$.

Example 3.2. (1) Define two binary operations "$\odot$" and "$\to$" on a set $A = \{1, a, b, c\}$ as follows:

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	a	b	c

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

Then, by routine calculation, we can see that $\mathbb{A} = (A; \odot, \rightarrow, 1)$ is a Hilbert semigroup.

(2) Define two binary operations "$\odot$" and "$\rightarrow$" on a set $X = \{1, a, b, c\}$ as follows:

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	b
c	1	1	b	c

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

Then, by routine calculation, we can see that $\mathbb{A} = (A; \odot, \rightarrow, 1)$ is a Hilbert semigroup.

Proposition 3.3. *Let* $\mathbb{A} = (A; \odot, \rightarrow, 1)$ *be a Hilbert semigroup. Then*

(i) $(\forall x \in A)$ $(1 \odot x = x \odot 1 = 1)$.

(ii) $(\forall x, y, z \in A)$ $(x \leq y \Rightarrow x \odot z \leq y \odot z,\ z \odot x \leq z \odot y)$.

Proof. (i) We know that $1 \odot x = (1 \rightarrow 1) \odot x = 1 \odot x \rightarrow 1 \odot x = 1$ and $x \odot 1 = x \odot (1 \rightarrow 1) = x \odot 1 \rightarrow x \odot 1 = 1$ for all $x \in A$.

(ii) Let $x, y, z \in A$ be such that $x \leq y$. Then

$$x \odot z \rightarrow y \odot z = (x \rightarrow y) \odot z = 1 \odot z = 1$$

and

$$z \odot x \rightarrow z \odot y = z \odot (x \rightarrow y) = z \odot 1 = 1.$$

Hence $x \odot z \leq y \odot z$ and $z \odot x \leq z \odot y$. □

Let $(A; \rightarrow, 1)$ be a Hilbert algebra and let $a, b \in A$. Then the set

$$A(a, b) := \{x \in A \mid a \rightarrow (b \rightarrow x) = 1\}$$

in nonempty because $a \rightarrow (b \rightarrow b) = a \rightarrow 1 = 1$ and/or $a \rightarrow (b \rightarrow 1) = 1$. If $A(a, b)$ has the least element, such element is uniquely determined by a and b. We denote it by $a + b$. If there exists $a + b$ for any a and b of a Hilbert algebra $(A; \rightarrow, 1)$, then we say that $(A; \rightarrow, 1)$ is an (H)-*Hilbert algebra* (see Hong and Jun [2]).

Proposition 3.4. *In an* (H)-*Hilbert algebra* $(A; \rightarrow, 1)$, *we have*

$$(\forall x, y, z \in A)\,((z + y) \rightarrow x = z \rightarrow (y \rightarrow x)).$$

Proof. Using (a4), (a5) and (a6), we have

$$y \leq (y \to x) \to x \leq (z \to (y \to x)) \to (z \to x) \\ = z \to ((z \to (y \to x)) \to x),$$

that is, $y \to (z \to ((z \to (y \to x)) \to x)) = 1$ which implies that

$$z + y \leq (z \to (y \to x)) \to x.$$

It follows from (a5) that $z \to (y \to x) \leq (z + y) \to x$. On the other hand, we obtain

$$z \leq y \to (z + y) \leq ((z + y) \to x) \to (y \to x)$$

by the definition of $+$ and (a6), which implies from (a5) that

$$(z + y) \to x \leq z \to (y \to x).$$

This completes the proof. □

Proposition 3.5. *Let* $\mathbb{A} = (A; \odot, \to, 1)$ *be a Hilbert semigroup in which* $(A; \to, 1)$ *is an* (H)*-Hilbert algebra. Then*

$$(\forall x, y, z \in A)((z \odot x) + (y \odot x) \leq (z + y) \odot x, \\ (x \odot z) + (x \odot y) \leq x \odot (z + y)).$$

Proof. Let $x, y, z \in A$. Using Proposition 3.4, we have

$$((z \odot x) + (y \odot x)) \to ((z + y) \odot x) = (z \odot x) \to ((y \odot x) \to ((z + y) \odot x)) \\ = (z \odot x) \to ((y \to (z + y)) \odot x) = (z \to (y \to (z + y))) \odot x = 1 \odot x = 1$$

and

$$((x \odot z) + (x \odot y)) \to (x \odot (z + y)) = (x \odot z) \to ((x \odot y) \to (x \odot (z + y))) \\ = (x \odot z) \to (x \odot (y \to (z + y))) = x \odot (z \to (y \to (z + y))) = x \odot 1 = 1.$$

Hence we have the desired results. □

Definition 3.6. An element $a (\neq 1)$ in a Hilbert semigroup $\mathbb{A} = (A; \odot, \to, 1)$ is said to be a *left* (resp. *right*) *unit divisor* if

$$(\exists b (\neq 1) \in A) \, (a \odot b = 1 \text{ (resp. } b \odot a = 1)).$$

A *unit divisor* is an element of A which is both a left and a right unit divisor.

Theorem 3.7. *Let $\mathbb{A} = (A; \odot, \rightarrow, 1)$ be a Hilbert semigroup. If it satisfies the left* (resp. *right*) *cancellation law for the operation $\odot$, that is,*

$$(\forall x(\neq 1), y, z \in A)\,(x \odot y = x \odot z \text{ (resp. } y \odot x = z \odot x) \;\;\Rightarrow\;\; y = z),$$

then A contains no left (resp. *right*) *unit divisors.*

Proof. Let $\mathbb{A} = (A; \odot, \rightarrow, 1)$ satisfy the left cancellation law for the operation $\odot$ and assume that $x \odot y = 1$, where $x \neq 1$. Then $x \odot y = 1 = x \odot 1$ by Proposition 3.3(i), which implies $y = 1$. Similarly for the right case, proving that there are no left (resp. right) unit divisors in A. □

Now we consider the converse of Theorem 3.7.

Theorem 3.8. *Let $\mathbb{A} = (A; \odot, \rightarrow, 1)$ be a Hilbert semigroup in which there are no left* (resp. *right*) *unit divisors. Then it satisfies the left* (resp. *right*) *cancellation law for the operation $\odot$.*

Proof. Let $x, y, z \in A$ be such that $x \odot y = x \odot z$ and $x \neq 1$. Then

$$x \odot (y \rightarrow z) = x \odot y \rightarrow x \odot z = 1$$

and

$$x \odot (z \rightarrow y) = x \odot z \rightarrow x \odot y = 1.$$

Since $\mathbb{A}$ has no left unit divisor, it follows that $y \rightarrow z = 1 = z \rightarrow y$ so that $y = z$. The argument is the same for the right case. □

Definition 3.9. Let $\mathbb{A} = (A; \odot, \rightarrow, 1)$ be a Hilbert semigroup. A nonempty subset D of A is called a *left* (resp. *right*) *deductive system* (LDS (resp. RDS), for short) if it satisfies:

(ds1) $A \odot D \subseteq D$ (resp., $D \odot A \subseteq D$).

(ds2) $(\forall a \in D)$ $(\forall x \in A)$ $(a \rightarrow x \in D \;\Rightarrow\; x \in D)$.

Example 3.10. Let $A = \{x, y, z, 1\}$ be a set with the following Cayley tables:

$\odot$	1	x	y	z
1	1	1	1	1
x	1	x	1	1
y	1	1	y	z
z	1	1	z	y

$\rightarrow$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	1	1	z
z	1	1	1	1

Then $\mathbb{A} = (A; \odot, \rightarrow, 1)$ is a Hilbert semigroup. We know that $D := \{x, 1\}$ is a LDS of $\mathbb{A}$, but $E := \{y, 1\}$ is not a LDS of $\mathbb{A}$ since $z \odot y = z \notin E$ and/or $y \rightarrow x = 1 \in E$, $y \in E$ but $x \notin E$.

Proposition 3.11. *If D is a LDS of a Hilbert semigroup $\mathbb{A} = (A; \odot, \rightarrow, 1)$, then*

$$(\forall a, b \in D)\,(A(a, b) \subseteq D).$$

Proof. Let $x \in A(a, b)$ where $a, b \in D$. Then $a \rightarrow (b \rightarrow x) = 1 \in D$ and so $x \in D$ by (ds2). Therefore $A(a, b) \subseteq D$. □

Theorem 3.12. *Let $\{D_i\}$ be an arbitrary collection of LDSs of a Hilbert semigroup $\mathbb{A} = (A; \odot, \rightarrow, 1)$, where i ranges over some index set. Then $\cap D_i$ is also a LDS of $\mathbb{A}$.*

Proof. Straightforward. □

Let $\mathbb{A} = (A; \odot, \rightarrow, 1)$ be a Hilbert semigroup. For any subset D of A, the intersection of all LDSs (resp. RDSs) of $\mathbb{A}$ containing D is said to be the *LDS* (resp. *RDS*) *generated by* D, and is denoted by $\langle D \rangle_l$ (resp. $\langle D \rangle_r$). It is clear that if D and E are subsets of a Hilbert semigroup $\mathbb{A} = (A; \odot, \rightarrow, 1)$ satisfying $D \subseteq E$, then $\langle D \rangle_l \subseteq \langle E \rangle_l$ (resp. $\langle D \rangle_r \subseteq \langle E \rangle_r$), and if D is a LDS (resp. RDS) of $\mathbb{A}$, then $\langle D \rangle_l = D$ (resp. $\langle D \rangle_r = D$).

Theorem 3.13. *Let $\mathbb{A} = (A; \odot, \rightarrow, 1)$ be a Hilbert semigroup and let D be a nonempty subset of A such that $A \odot D \subseteq D$. Then*

$$\langle D \rangle_l := \{a \in A | y_n \rightarrow (\cdots \rightarrow (y_1 \rightarrow a) \cdots) = 1 \text{ for some } y_1, \cdots, y_n \in D\}.$$

Proof. Denote

$$\Omega := \{a \in A | y_n \rightarrow (\cdots \rightarrow (y_1 \rightarrow a) \cdots) = 1 \text{ for some } y_1, \cdots, y_n \in D\}.$$

Let $a \in A$ and $b \in \Omega$. Then there exist $y_1, \cdots, y_n \in D$ such that

$$y_n \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots) = 1.$$

It follows that

$$\begin{aligned} 1 = x \odot 1 &= x \odot (y_n \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots)) \\ &= x \odot y_n \rightarrow (\cdots \rightarrow (x \odot y_1 \rightarrow x \odot b) \cdots). \end{aligned}$$

Since $x \odot y_i \in D$ for $i = 1, \cdots, n$, we have $x \odot b \in \Omega$, i.e., $A \odot \Omega \subseteq \Omega$. Let $x, a \in A$ be such that $a \to x \in \Omega$ and $a \in \Omega$. Then there exist $y_1, \cdots, y_n, z_1, \cdots, z_m \in D$ such that

$$y_n \to (\cdots \to (y_1 \to (a \to x)) \cdots) = 1\,, \tag{1}$$

$$z_m \to (\cdots \to (z_1 \to a) \cdots) = 1\,. \tag{2}$$

Using (a5), it follows from (1) that

$$a \to (y_n \to (\cdots \to (y_1 \to x) \cdots)) = 1,$$

i.e., $a \leq y_n \to (\cdots \to (y_1 \to x) \cdots)$ so from (2) and (a7) that

$$\begin{aligned} 1 &= z_m \to (\cdots \to (z_1 \to a) \cdots) \\ &\leq z_m \to (\cdots \to (z_1 \to (y_n \to (\cdots \to (y_1 \to x) \cdots))) \cdots). \end{aligned}$$

Thus $z_m \to (\cdots \to (z_1 \to (y_n \to (\cdots \to (y_1 \to x) \cdots))) \cdots) = 1$, which implies $x \in \Omega$. Therefore Ω is a LDS of $\mathbb{A}$. Obviously $D \subseteq \Omega$. Let G be a LDS containing D. To show $\Omega \subseteq G$, let a be any element of Ω. Then there exist $y_1, \cdots, y_n \in D$ such that

$$y_n \to (\cdots \to (y_1 \to a) \cdots) = 1 \in G.$$

It follows from (ds2) that $a \in G$ so that $\Omega \subseteq G$. Consequently, we have $\langle D \rangle_l = \Omega$. □

We know that, in the following example, the union of any LDSs (resp. RDSs) D and E may not be a LDS (resp. RDS) of a Hilbert semigroup $\mathbb{A} = (A; \odot, \to, 1)$.

Example 3.14. Let $A = \{a, b, c, d, 1\}$ be a set with the following Cayley tables:

$\odot$	1	a	b	c	d
1	1	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	d

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	1	b	b	1

Then $\mathbb{A} = (A; \odot, \to, 1)$ is a Hilbert semigroup. We know that $D := \{a, 1\}$ and $E := \{b, 1\}$ are LDSs of $\mathbb{A}$, but $D \cup E = \{a, b, 1\}$ is not a LDS of $\mathbb{A}$, since $b \to c = a \in D \cup E$, $c \notin D \cup E$.

Theorem 3.15. *Let D and E be LDSs of a Hilbert semigroup $\mathbb{A} = (A; \odot, \rightarrow, 1)$. Then*

$$\langle D \cup E\rangle_l := \left\{ a \in A \;\middle|\; \begin{array}{l} x \rightarrow (y \rightarrow a) = 1 \\ \text{for some } x \in D \text{ and } y \in E \end{array} \right\}.$$

Proof. Denote

$$K := \left\{ a \in A \;\middle|\; \begin{array}{l} x \rightarrow (y \rightarrow a) = 1 \\ \text{for some } x \in D \text{ and } y \in E \end{array} \right\}.$$

Obviously, $K \subseteq \langle D \cup E\rangle_l$. Let $b \in \langle D \cup E\rangle_l$. Then there exist $y_1, \cdots, y_n \in D \cup E$ such that

$$y_n \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots) = 1$$

by Theorem 3.13. If $y_i \in D$ (resp. E) for all $i = 1, \cdots, n$, then $b \in D$ (resp. E). Hence $b \in K$ since $b \rightarrow (1 \rightarrow b) = 1$ (resp. $1 \rightarrow (b \rightarrow b) = 1$). If some of $y_1, \cdots, y_n$ belong to D and others belong to E, then we may assume that $y_1, \cdots, y_k \in D$ and $y_{k+1}, \cdots, y_n \in E$ for $1 \leq k < n$, without loss of generality. Let $p = y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots)$. Then

$$\begin{aligned} & y_n \rightarrow (\cdots \rightarrow (y_{k+1} \rightarrow p) \cdots) \\ & \quad = y_n \rightarrow (\cdots \rightarrow (y_{k+1} \rightarrow (y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots))) \cdots) \\ & \quad = 1, \end{aligned}$$

and so $p \in E$. Now let $q = p \rightarrow b = (y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots)) \rightarrow b$. Then

$$\begin{aligned} & y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow q) \cdots) \\ & \quad = y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow ((y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots)) \rightarrow b)) \cdots) \\ & \quad = (y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots)) \rightarrow (y_k \rightarrow (\cdots \rightarrow (y_1 \rightarrow b) \cdots)) \\ & \quad = 1, \end{aligned}$$

which implies that $q \in D$. Since $p \rightarrow (q \rightarrow b) = q \rightarrow (p \rightarrow b) = q \rightarrow q = 1$, it follows that $b \in K$ so that $\langle D \cup E\rangle_l \subseteq K$. This completes the proof. □

References

[1] D. Busneag, A note on deductive systems of a Hilbert algebra, *Kobe J. Math.*, **2** (1985), 29-35.

[2] S.M. Hong, Y.B. Jun, On a special class of Hilbert algebras, *Algebra Colloq.*, **3**, No. 3 (1996), 285-288.

[3] Y.B. Jun, Deductive systems of Hilbert algebras, *Math. Jpn.*, **43**, No. 1 (1996), 51-54.