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SINGULAR CURVES AND RANK ONE TORSION FREE SHEAVES

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Abstract: Here we introduce and study a few notions for spanned rank one torsion free sheaves on singular projective vurves.

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1. Singular Curves and Rank One Torsion Free Sheaves

Let A be a smooth and connected quasi-projective curve. Fix an integer $s \geq 1$, s distinct points $Q_1, \dots, Q_s \in A$ and integers $a_i > 0$, $1 \leq i \leq s$. Let m_{A,Q_i} be the maximal ideal of the local ring $\mathcal{O}_{A,Q_i}$. There is a unique integral quasi-projective curve B, $P \in B$ and $f : A \to B$ such that $P = \mathrm{Sing}(B)$, $f^{-1}(P) = \{Q_1, \dots, Q_s\}$, $f_{|A\backslash\{Q_1,\dots,Q_s\}} : A\backslash\{Q_1, \dots, Q_s\} \to B\backslash\{P\}$ is an isomorphism and f^* induces an isomorphism between the maximal ideal $m_{B,P}$ of the local ring $\mathcal{O}_{B,P}$ and the ideal $\prod_{i=1}^{s} {m_{A,Q_i}}^{a_i}$ of the semilocal ring $\oplus_{i=1}^{s}\mathcal{O}_{A,Q_i}$ ([2]). Indeed, with the terminology of [2] the singularity (B, P) is said to have a modulus; it is associated to the effective divisor $a_1Q_1 + \cdots + a_sQ_s$.

Definition 1. Let X be an integral projective curve, $u : C \to X$ its normalization, $\mathcal{F}$ a rank one torsion free sheaf on X and $V \subseteq H^0(X, \mathcal{F})$ a

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linear subspace. Set $n := \dim(V) - 1$. Since C is smooth and one-dimensional, the torsion free sheaf $u[\mathcal{F}] := u^*(\mathcal{F})/\mathrm{Tors}(u^*(\mathcal{F}))$ is locally free. Since tensor profduct is a right tensor function, it is easy to check that the image $u[V]$ of $u^*(V)$ in $H^0(C, u[\mathcal{F}])$ spans $u[\mathcal{F}]$. Looking at the dense open set $u^{-1}(X_{reg})$ we easily get $\dim(u[V]) = \dim(V) = n+1$. Hence the pair $(u[\mathcal{F}], u[V])$ defines a morphism $w : C \to \mathbf{P}^n$ such that $w(C)$ spans $\mathbf{P}^n$. By the universal property of singularities with a modulus, there is a unique (up to isomorphisms) integral curve Y and morphisms $a : C \to Y$, $a' : Y \to X$, $f : Y \to \mathbf{P}^n$ such that:

(i) Y has only singularities with a modulus, $u = a' \circ a$ and $u = f \circ a$;

(ii) for every integral curve Y_1 whose only singularity such that there are morphisms $a_1 : C \to Y_1$, $a_1' : Y_1 \to X$, $f_1 : Y_1 \to \mathbf{P}^n$ with $u = a_1' \circ a_1$ and $u = f_1 \circ a_1$ there is a morphism $b : Y_1 \to Y$ such that $a = b \circ a_1$.

We will say that the quadruple (Y, a, a', f) is the modular minimal model of the triple $(X, \mathcal{F}, V)$. If there is a similar defined quadruple (Y, a, a', f) in which we drop the assumption that Y and Y_1 have only singularities with a modulus, then we will call this quadruple the universal minimal model of $(X, \mathcal{F}, V)$.

Definition 2. Let X be an integral projective curve and $\mathcal{F}$ a rank one torsion free sheaf on X. We will say that $\mathcal{F}$ is weakly full if there are an integral projective curve Y, a birational morphism $b : Y \to X$ and $L \in \mathrm{Pic}(Y)$ such that $\mathcal{F} \cong b_*(L)$. Notice that $h^0(X, \mathcal{F}) = h^0(Y, L)$ and $\deg(\mathcal{F}) = \deg(L) + p_a(X) + p_a(Y)$ and hence $h^1(X, \mathcal{F}) = h^1(Y, L) + p_a(X) - p_a(Y)$ (Riemann-Roch). Now assume $\mathcal{F}$ spanned. Hence L is spanned and if $V \subseteq H^0(X, \mathcal{F})$ is a linear subspace of $H^0(X, \mathcal{F})$ spanning $\mathcal{F}$, then the associated $\dim(V)$-dimensional linear subspace $b[V]$ of $H^0(Y, L)$ spans L.

Our first main result is that the universal minimal model always exists and to give an explicit construction of it.

Remark 1. Let X be an integral projective curve and $u : C \to X$ its normalization. Fix a spanned rank one torsion free sheaf $\mathcal{F}$ on X and an $(n+1)$-dimensional, $n \geq 1$, linear subspace V of $H^0(X, \mathcal{F})$ spanning $\mathcal{F}$. See V as a linear subspace $u[V]$ of $H^0(C, u[\mathcal{F}])$ spanning $u[\mathcal{F}]$ and call $f : C \to \mathbf{P}^n$ the morphism induced by the pair $(u[\mathcal{F}], u[V])$. Since u is birational, the map $(u, f) : C \to X \times \mathbf{P}^n$ is birational onto its image. Set $Y := \mathrm{Im}((u, f))$ and let $a := (u, f)$ seen as a birational morphism $C \to Y$. The projections from Y into the two factors of $X \times \mathbf{P}^n$ induces morphisms $a' : Y \to X$, $f : Y \to \mathbf{P}^n$ such that $u = a' \circ a$ and $u = f \circ a$. It is easy to check that the the quadruple (Y, a, a', f) is the universal minimal model of the triple $(X, \mathcal{F}, V)$. From the normalization map $C \to Y$ it is obiovous how to obtain the universal modular reduction of the triple $(X, \mathcal{F}, V)$: use the universal property of singularities with a modulus.

Let D be any integral projective curve and $\mathcal{F}$ any rank one torsion free sheaf on D. Set $\mathrm{Sing}(\mathcal{F}) := \{P \in D : \mathcal{F}$ is not locally free at $P\}$. Hence $\mathrm{Sing}(\mathcal{F}) \subseteq \mathrm{Sing}(D)$.

Definition 3. Let T be an integral projective curve and $h : T \to \mathbf{P}^n$, $n \geq 1$, a non-degenerate morphism. For each $P \in T_{reg}$ let $a_P \geq 0$ be the order of vanishing of the differential $d(h)$ of h at P. Let $\gamma : T \to Y$ be the only birational and bijective morphism between integral curves which is a local isomorphism at each point of $\mathrm{Sing}(T)$ and such that each point $\gamma(Q)$, $Q \in T_{reg}$, is a unibranch singularity with a modulus in the sense of [2], with T as partial normalization and with modulus $a_Q + 1$ at Q. By the universal property of the singularities with a modulus there is a unique morphism $f : Y \to \mathbf{P}^n$ such that $h = f \circ \gamma$. We will say that the pair (Y, f) is the cuspidal reduction of the pair (T, h).

Proposition 1. *Fix integers $k \geq 2$ and $x > 0$ and a set $S \subset \mathbf{P}^1$ such that $\sharp(S) = kx$. Let $h : C \to \mathbf{P}^1$ be the degree k cyclic covering of $\mathbf{P}^1$ ramified exactly on S. Hence C is a genus $(1 + k(k-1)x/2 + k)$ smooth curve. Let $f : Y \to \mathbf{P}^1$ be the cuspidal reduction of h. Then $p_a(Y) = (k-1)kx + p_a(C) = 3(k-1)kx/2 + k + 1$ and $f_*(\mathcal{O}_Y) \cong \oplus_{i=0}^{k-1} \mathcal{O}_{\mathbf{P}^1}(-3x)$.*

Proof. The equality $p_a(Y) = (k-1)kx + p_a(C)$ follows from the structure of h and the definition of cuspidal reduction. Set $R := \mathcal{O}_{\mathbf{P}^1}(-x)$ and $M := \mathcal{O}_{\mathbf{P}^1}(-3x)$. Since h is a cyclic covering of $\mathbf{P}^1$ ramified at exactly kx points of S, we have $h_*(\mathcal{O}_C) \cong \oplus_{i=0}^{k-1} R^{\otimes i}$ and the pair (C, h) is uniquely determined by the choice of R and the set S. Since $h^i(C, \mathcal{O}_C) = h^i(\mathbf{P}^1, h_*(\mathcal{O}_C))$ by the finiteness of h we obtain $p_a(C) = 1 + k(k-1)x/2 + k$ (or use the Riemann-Hurwitz formula). The torsion-free sheaf $f_*(\mathcal{O}_Y)$ is locally free because $\mathbf{P}^1$ is a smooth curve. Since cyclic group $\mathbb{Z}/k\mathbb{Z}$ act on the pair (C, h), it acts on the pair (Y, f) and hence on the rank k vector bundle $f_*(\mathcal{O}_Y)$. Since this group is cyclic, we easily get $f_*(\mathcal{O}_Y) \cong \oplus_{i=0}^{k-1} L^{\otimes i}$ for some $L \in \mathrm{Pic}(\mathbf{P}^1)$. For degree reason we get $L \cong M$. □

Notation 1. For all integers g, r, d set $\rho(g, r, d) := g - (r+1)(g + r - d)$ (the Brill-Noether number).

Theorem 1. *Fix integers $q > 0$ and $g \geq 2q + 1$ and a smooth and connected genus q projective curve C. If $\mathrm{char}(\mathbb{K}) = 2$ assume $2q + 1 \leq g \leq 3q$. Then there exists an integral projective curve X with C as its normalization, with exactly $g - q$ ordinary cusps as only singularities and $L \in \mathrm{Pic}(X)$ such* *that L is spanned by its global sections* $\deg(L) > 0$ *(or, equivalently (by the other assumptions)* $h^0(X, L) \geq 2$*) and* $\rho(g, \deg(L), 1) < 0$.

Proof. First assume either $\mathrm{char}(\mathbb{K}) \neq 2$ and $2q+1 \leq g \leq 5q$ or $\mathrm{char}(\mathbb{K}) = 2$ and $2q+1 \leq g \leq 3q$. Notice that $\rho(g, q+1, 1) < 0$ because $g \geq q+1$. Counting dimension we see the existence of $M \in \mathrm{Pic}^{q+1}(C)$ such that $h^1(C, M) = 0$ (i.e. $h^0(C, M) = 2$), M is spanned and the induced morphism $h_M : C \to \mathbf{P}^1$ has only ordinary ramification points (even if $\mathrm{char}(\mathbb{K}) = 2$, i.e. in this case each of them count only two for the Riemann-Hurwitz formula of h_M). By Riemann-Hurwitz h_M has exactly $4q$ (resp. $2q$) ramification points if $\mathrm{char}(\mathbb{K}) \neq 2$ (resp. $\mathrm{char}(\mathbb{K}) = 2$). Fix any $g - q$ of them and call X the curve with C as normalization, $g - q$ ordinary cusp as only singularities and these cusps have as counterimages in C exactly these marked points. Hence h_M induces a degree $q + 1$ morphism $f : X \to \mathbf{P}^1$ and hence $L \in \mathrm{Pic}^{q+1}(X)$ such that $h^0(X, L) \geq 2$, proving this case. Now assume $\mathrm{char}(\mathbb{K}) \neq 2$ and $g > 5q$. There is an integer $x \geq q + 2$ such that $3q - 2 + 2x \geq g$. Fix any such integer x. Counting dimensions we see the existence $M \in \mathrm{Pic}^{q+1}(C)$ such that $h^1(C, M) = 0$ (i.e. $H^0(C, L) = x - q + 1$), M is spanned and a general two-dimensional linear subspace V of $H^0(C, M)$ induces morphism $h_V : C \to \mathbf{P}^1$ has only ordinary ramification points. Since h_V has exactly $2q-2+2x$ ramification points (Riemann-Hurwitz) and $\rho(g, x, 1) < 0$ because $g \geq 2x - 1$, we may repeat without any modification the proof of the case $2q + 1 \leq g \leq 5q$. □

Proposition 2. *Fix integers $q > 0$ and g and a smooth and connected genus q projective curve C. Assume either q even and $g = q+1$ or* $\mathrm{char}(\mathbb{K}) = 0$ *and $g = q + 2$. Then there exists an integral projective curve X with C as its normalization, with exactly $g - q$ ordinary cusps as only singularities and $L \in \mathrm{Pic}(X)$ such that L is spanned by its global sections* $\deg(L) > 0$ *(or, equivalently (by the other assumptions)* $h^0(X, L) \geq 2$*) and* $\rho(g, \deg(L), 1) < 0$.

Proof. Let k be the minimal integer such that there is $M \in \mathrm{Pic}^k(C)$ such that $h^0(C, M) \geq 2$. By the minimality of k we have $h^0(C, M) = 2$ and M is spanned by its global sections. Hence M induces a degree k morphism $h_M : C \to \mathbf{P}^1$. If $\mathrm{char}(\mathbb{K}) = 0$, then $\mathbb{A}^1_{\mathbb{K}}$ has no étale covering and hence the differential of h_M vanishes at at least two points of C. By Brill-Noether theory we have $k \leq \lfloor (q + 3)/2 \rfloor$. Hence $\rho(q + 2, k, 1) < 0$ and $\rho(q + 1, k, 1) < 0$ if q is even. In arbitrary characteristic $\mathbf{P}^1_{\mathbb{K}}$ has no étale covering and hence in the same way we get the case $g = q + 1$ and q even. □

Remark 2. Proposition 2 is false for $q = 0$ (see [1]).

Proposition 3. *Fix integers $q \geq 2$ and g. If $\mathrm{char}(\mathbb{K}) \neq 2$ and q is even assume $q+1 \leq g \leq 4q$. If $\mathrm{char}(\mathbb{K}) \neq 2$ and q is odd assume $q+2 \leq g \leq 4q+1$. If $\mathrm{char}(\mathbb{K}) = 2$ and q is even assume $q+1 \leq g \leq 5q/2$. If $\mathrm{char}(\mathbb{K}) = 2$ and q is odd assume $q+2 \leq g \leq 2q+(q+1)/2$. Let C be a general smooth curve of genus q. Then there exists an integral projective curve X with C as its normalization, with exactly $g-q$ ordinary cusps as only singularities and $L \in \mathrm{Pic}^k(X)$, $k := \lfloor (q+3)/2 \rfloor$, such that L is spanned by its global sections, and $h^0(X,L) = 2$ and $\rho(g, \deg(L), 1) < 0$.*

Proof. There is $M\mathrm{Pic}^k(C)$, $k := \lfloor (q+3)/2 \rfloor$ such that $h^0(C,M) = 2$ and M is spanned by its global sections and the induced degree k morphism has only ordinary ramification points (even if $\mathrm{char}(\mathbb{K}) = 2$). Apply the Riemann-Hurwitz formula. □

Proposition 4. *Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Fix integers q, g, k such that $q \geq 3$, $2 \leq k < lfloor(q+3)/2\rfloor$ and $q+1 \leq g \leq 3q-2+2k$. Let C be a general smooth k-gonal curve of genus q. Let C be a general smooth curve of genus q. Then there exists an integral projective curve X with C as its normalization, with exactly $g-q$ ordinary cusps as only singularities and $L \in \mathrm{Pic}^k(X)$ such that L is spanned by its global sections, and $h^0(X,L) = 2$ and $\rho(g, \deg(L), 1) < 0$.*

Proof. It is known that the degree k morphism $C \to \mathbf{P}^1$ has only ordinary ramification points. Apply the proofs of Proposition 3 and Proposition 4. □

Let F_e, $e \geq 0$, the Hirzebruch surface with invariant e, i.e. such that $-e$ is the minimal self-intersection of a section of the ruling of F_e. Hence $F_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ is isomorphic to a smooth quadric surface and F_1 is isomorphic to the blowing-up of $\mathbf{P}^2$ at one point. We fix a ruling $u : F_e \to \mathbf{P}^1$; f is unique if and only if $e > 0$. We have $\mathrm{Pic}(F_e) \cong \mathbf{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(F_e)$ asection h of f with $h^2 = -e$ and a class, f, of the ruling u. Thus $h \cdot f = 1$ and $f^2 = 0$. We will use both the additive and the multiplicative notation for line bundles and divisors on F_e. We have $\omega_{F_e} \cong -2h-(2+e)F$. We have $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a < 0$. By the projection formula we have $f_*(\mathcal{O}_{F_e}(ah+bf)) \cong \oplus_{i=0}^{a} \mathcal{O}_{\mathbf{P}^1}(b-ie)$ for every $a \geq 0$. Thus $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \geq 0$ and $b < ea$, $h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = \sum_{i=1}^{a}(b-ie+1) = (2b+2-ae)(a+1)/2$ if $a \geq 0$ and $b \geq 0$ and $h^1(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if $a \geq 0$ and $b \geq ae-1$. Now we will see how to translate any statement concerning the postulation of a zero-dimensional scheme $Z \subset F_e$ in a statement concerning an interpolation problem for suitable polynomials.

Notation 2. First assume $e = 0$ and fix integers $a \geq 0$ and $b \geq 0$. We may parametrize $H^0(F_0, \mathcal{O}_{F_0}(ah + bf))$ as the linear subspace of $\mathbb{K}[w_0, w_1, x_0, x_1]$ of all polynomials spanned by the monomials with degree a in the variables w_0, w_1 and degree b in the variables x_0, x_1. Fix a reduced $T \in |ah + bf|$ and $P \in T$; we will say that P is a cuspidal point if the normalization map of T is not étale over P; we will say that a cuspidal point (resp. an ordinary cusp) $P \in C$ is a vertical cuspidal point if it is a cuspidal point (resp. an ordinary cusp) and the line of type $(1, 0)$ through P is tangent at P to a non-ordinary branch of T at P. Now assume $e > 0$. Fix integers $a > 0$ and $b \geq ae$. In the polynomial ring $\mathbb{K}[w, x_0, x_1]$ assign weight one to the variables x_0, x_1. We may parametrize $H^0(F_0, \mathcal{O}_{F_0}(ah + bf))$ as the linear subspace of $\mathbb{K}[w, x_0, x_1]$ with total weight b and with degree at most a in the variable w. Fix a reduced $T \in |ah + bf|$ and $P \in T$; we will say that P is a cuspidal point if the normalization map of T is not étale over P; we will say that a cuspidal point (resp. an ordinary cusp) $P \in T$ is a vertical cuspidal point if it is a cuspidal point (resp. an ordinary cusp) and the fiber of u passing through P is tangent at P to a non-ordinary branch of T at P.

Remark 3. Take the set-up of Notation 2, $T \in |ah + bf|$ and $P \in T$ a vertical cuspidal point. First assume $e > 0$. Let $m(w, x_0, x_1)$ be an equation of C. First assume that $\partial_w(m)$ is not identically zero. Notice that $\partial_w(m)$ is weighted homogeneous of weight $b - e$ and each monomial appearing with non-zero coefficient in its expansion has degree at most $a - 1$ in the variable w. Hence $\partial_w(m)$ defines a curve $T_w \in |(a-1)h + (b-e)f|$. By assumption the intersection number of T and T_w at P is at least 3. Notice that $T_w \cdot T = (ah + bf) \cdot ((a-1)h + (b-e)f) = b(a-1) + (b-e)a - ea(a-1)$. Hence T has at most $\lfloor (2ba - b - ea^2)/3 \rfloor$ vertical cuspidal points. Now assume $\partial_w(m) \equiv 0$. If $\mathrm{char}(\mathbb{K}) = 0$,then this implies that m does not depend from w. Since $a > 0$, this easily implies that T has multiple components, contradiction. Now assume $\partial_w(m) \equiv 0$ and $p := \mathrm{char}(\mathbb{K}) > 0$. We get that in each monomial appearing with a non-zero coefficient of m the variable w appears to a p-power. As in the characteristic zero case this gives a contradiction if $p > a$. Now assume $e = 0$ and take an equation $m(w_0, w_1, x_0, x_1)$ of C. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > a$. We obtain that at least one of the two partial derivatives $\partial_{w_0}(m)$ or $\partial_{w_1}(m)$ is non-zero and hence it defines a curve $T' \in |(a-1)h + bf|$. As above we get that T has at most $\lfloor (2ab - b)/3 \rfloor$ vertical cuspidal points.

In summary we proved the following results.

Proposition 5. *Fix integers $e > 0$, $a > 0$ and $b \geq ea$ and an integral curve $T \subset F_e$ such that $T \in |ah + bf|$. Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > a$.

Then T has at most $\lfloor (2ba - b - ea^2)/3 \rfloor$ vertical cuspidal points.

Proposition 6. *Fix integers $a > 0$, $b > 0$ and an integral curve $T \subset F_e$ such that $T \in |ah + bf|$. Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > a$. *Then T has at most $\lfloor (2ab - b)/3 \rfloor$ vertical cuspidal points.*

These results are two weak to get lower bounds for the scrollar invariants of the cuspidal reduction of morphisms $C \to \mathbf{P}^1$. We hope that some reader may do better.

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