International Journal of Pure and Applied Mathematics
Volume 19 No. 1 2005, 67-85

WAVE DIFFRACTION BY CIRCULAR AND ELLIPTICAL CYLINDERS IN FINITE DEPTH WATER

Dambaru D. Bhatta

Department of Mathematics
University of Texas-Pan American
1201 West University Drive, Edinburg, TX 78541, USA
e-mail: bhattad@utpa.edu

Abstract: Here we consider the linear wave diffraction due to cylindrical structures. First we formulate the diffraction problem for a large, fixed, vertical, bottom-mounted, surface-piercing cylindrical structure in water of finite depth. The analytical solution for circular cylinder is obtained in terms of Bessel and Hankel functions using the method of separation of variables. The diffraction problem for elliptical cylinder is solved using elliptical coordinates and separation of variables method. The analytical solution is obtained in terms of Mathieu and modified Mathieu functions. Also we discuss how to obtain the analytical expressions for the wave forces on those structures. As a limiting case, we show two different ways to derive the velocity potential function for circular cylinder from the velocity potential function of elliptical cylinder.

AMS Subject Classification: 76B15, 35Q35, 35J05
Key Words: wave diffraction, cylindrical structures, Mathieu function

1. Introduction

To design an offshore structure, the evaluation of the hydrodynamic coefficients and wave loads on the structure is an important task. The forces exerted by surface waves on offshore structures such as offshore drilling or submerged

Received: January 21, 2005

oil storage tanks play a significant role in the design of large submerged or semi-merged structures. It is not an easy task to evaluate these loads since it involves the complexity of the interaction of waves with the structure. Due to randomness of the waves, even nonlinear wave theories may not be adequate to describe the wave-structure interactions. However, some of the existing theories coupled with our understanding of the interaction phenomenon through analytical studies, laboratory experiments and at-sea measurements are accurate enough to predict wave forces on various offshore structures. Wave forces are computed using different formulations depending on the type and size of the members in an offshore structure. The wave forces are calculated from the pressure distribution which is obtained from the velocity potential function.

When the structure is large compared to the wavelength, the incident waves upon arriving at the structure undergo significant scattering or diffraction. In this case the diffraction of waves from the structure should be taken into account in calculation of the wave forces. Solution to the linear wave diffraction problem is available for an isolated surface-piercing vertical circular cylinder extending from the seabed. This was first treated by MacCamy and Fuchs [8] for arbitray depth. The diffraction theory for circular cylinders formulated by MacCamy and Fuchs has been extensively used by a number of investigators to predict the wave loads on large submerged circular cylinders. Garrett [7] presented the results of the scattering of surface waves by a circular dock. Dean and Dalrymple [4] presented a review of potential flow hydrodynamics. They discussed the formulation of the linear water wave theory and development of the simplest two-dimensional solution for standing and progressive waves. Chau and Eatock-Taylor [3] provided a detailed analysis of the second order diffraction problem of a uniform vertical circular cylinder in regular waves. Rahman and Bhatta [11] have obtained closed form analytical solutions for the added mass and damping coefficients due to an oscillating cylinder in waves. Debnath [6] presented mathematical theory of nonlinear water waves with applications. He studied the theory of nonlinear shallow water waves and solitons, with emphasis on methods and solutions of several evolution equations that are originated in the theory of water waves. Rahman [10] discussed the mathematical and physical aspects of the theory of water waves. Bhatta and Rahman [2] have presented anlytical expressions for the wave forces and the hydrodynamic coefficients due to the diffraction and radiation by a floating, vertical, circular cylinder in water of finite depth.

To evaluate the wave forces, we first need to evaluate the velocilty potential functions, due to incident and diffracted waves. In this paper, we consider water wave diffraction due to large, vertical, surface piercing cylindrical structures, circular and elliptical. Then we solve these two diffraction problems by using the method of separation of variables. The analytical solution for the circular cylinder is obtained in terms of Bessel and Hankel functions. To solve the diffraction problem for an elliptical cylinder, we use elliptical coordinates. The analytical solution is obtained in terms of Mathieu and modified Mathieu functions. Also we mention how to derive the analytical expressions for the wave forces on those structures. As a limiting case, we derive the velocity potential function for circular cylinder case from the velocity potential function of elliptical cylinder case. We derive this by two seperate methods. One way to achieve this, first we convert the governing Mathieu equation to a Bessel equation as the limiting case. Then the solution can be written in terms of Bessel function and Hankel function. Another method we use here is the degenerate form of the Mathieu functions and modified Mathieu functions [8].

2. Mathematical Formulation

We consider a surface gravity wave of amplitude A by a large, vertical, surface piercing, cylindrical structure in water of finite depth h. The structure is extending from sea-bed. Let (x, y, z) be the Cartesian coordinate system with x measured in the direction of wave propagation and z vertically upwards from still water level and coincide with the axis of the cylinder. The flow is assumed to be incompressible, irrotational and oscillatory.

For an incompressible fluid, the equation of continuity is

$$\operatorname{div}(\vec{\mathbf{v}}) \quad = \quad 0\,, \tag{1}$$

where $\vec{\mathbf{v}} = (u, v, w)$ is the velocity of the fluid. Considering the fluid to be inviscid, the Euler equation of motion can be expressed as

$$\frac{D\vec{\mathbf{v}}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{\mathbf{F}}\,, \tag{2}$$

where $\vec{\mathbf{F}}(\vec{\mathbf{x}}, t)$ is the external or body force per unit mass, ρ is the density and p is the pressure of the fluid. In case of water waves, the body force which is almost always present is gravity. So we can write $\vec{\mathbf{F}} = -g\vec{\mathbf{k}}$, where g is the acceleration due to gravity and $\vec{\mathbf{k}}$ is the unit vector in the positive z-direction. Thus, the equations of motion become

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,, \tag{3}$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$.

Assumption of irrotationality gives us curl$(\vec{\mathbf{v}}) = 0$, so that we can write $\vec{\mathbf{v}}$ =grad(Φ). Now the equation of continuity (1) becomes

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0\,, \tag{4}$$

i.e., Φ satisfies Laplace equation in the fluid region.

The dynamic and kinematic free surface boundary conditions, respectively, are

$$\begin{aligned} \frac{\partial\Phi}{\partial t} + g\zeta + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] &= 0 \text{ on } z = \zeta\,, && (5)\\ \frac{\partial\zeta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\zeta}{\partial y} - \frac{\partial\Phi}{\partial z} &= 0 \text{ on } z = \zeta\,. && (6) \end{aligned}$$

Here $\zeta = \zeta(x, y)$ is the free surface elevation function. Considering the linear wave diffraction problem, the conditions (5) and (6) can be combined into one condition as

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \text{ on } z = 0\,. \tag{7}$$

The boundary conditions at the ocean bottom ($z = -h$) and at the body surface under the water, S_B , respectively, are:

$$\begin{aligned} \frac{\partial\Phi}{\partial z} &= 0 \text{ at } z = -h\,, && (8)\\ \frac{\partial\Phi}{\partial n} &= 0 \text{ on } S_B\,. && (9) \end{aligned}$$

At a given point in the fluid region at time t, this velocity potential, Φ, consists of the velocity potential due to the incident waves, Φ_I, and the velocity potential due to the waves scattered or diffracted from the surface of the structure, Φ_S. Then

$$\Phi = \Phi_I + \Phi_S\,, \tag{10}$$

in which Φ, Φ_I, and Φ_S are functions of x, y, z, and t. The incident velocity potential, Φ_I, satisfies the boundary conditions outlined above in the absence of the structure. The body surface boundary condition may be written as

$$\frac{\partial\Phi_S}{\partial n} = -\frac{\partial\Phi_I}{\partial n} \text{ on } S_B\,. \tag{11}$$

Also Φ_S should satisfy the radiation condition at infinity. Assuming a plane wave of amplitude A with frequency σ and wave number k incident on the cylinder, the velocity potential $\Phi(r,\theta,z,t)$ can be written as

$$\Phi(x,y,z,t) = \mathrm{Re}\left[\phi(x,y,z)\, e^{-i\sigma t}\right], \tag{12}$$

where *Re* stands for the real part, $\phi(x,y,z)$ is the complex potential and can be written as the sum of incident and scattered wave potentials as follows $\phi(x,y,z) = \phi_I(x,y,z) + \phi_S(x,y,z)$. Also ϕ satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0\,. \tag{13}$$

This completes the statement of the boundary value problem.

3. Solution for a Circular Cylinder

Here we take the cylindrical structure to be circular and assume that the radius of this cylinder is b. We consider the cylindrical coordinate system (r, θ, z) with r measured radially from $z-$axis and θ from the positive $x-$axis. In this case the governing equation (13) in the fluid region becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0\,. \tag{14}$$

The free surface condition, bottom boundary condition and body surface condition respectively are as follows:

$$\sigma^2\phi - g\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z=0, \quad r \geq b\,, \tag{15}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z=-h, \quad r \geq b\,, \tag{16}$$

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{on} \quad r=b, \quad -h < z < 0\,. \tag{17}$$

Now we use the method of separation of variables (Debnath [6]) to solve this boundary value problem. So we assume a solution of the form

$$\phi(r,\theta,z) = R(r)Z(z)\Theta(\theta)\,. \tag{18}$$

Then equation (14) yields three ordinary differential equations

$$\frac{d^2Z}{dz^2} - k^2 Z = 0\,, \tag{19}$$

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0\,, \tag{20}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)R = 0\,, \tag{21}$$

where k^2 and m^2 are constants of separation. Due to symmetry, solution of the equation (20) contains terms involving $\cos m\theta$ only. Using the conditions (15) and (16), the differential equation (19) gives us

$$Z(z) = C\frac{\cosh k(z+h)}{\cosh kh}\,, \tag{22}$$

where k satisfies the dispersion relation

$$\sigma^2 = gk\tanh kh. \tag{23}$$

The constant C can be determined using the proper boundary conditions. Equation (21) is Bessel's differential equation. The Bessel function of the first kind of order m, $J_m(kr)$, satisfies this equation, and thus this will constitute a solution to the incident wave potential. Bessel function of third kind of order m, $H_m^{(1)}(kr)$ (Hankel function of first kind of order m), satisfies equation (21). For large argument Hanklel function takes the following form

$$H_m^{(1)}(\varsigma) \approx \sqrt{\frac{2}{\pi\varsigma}}e^{i\left(\varsigma - \frac{m\pi}{2} - \frac{\pi}{4}\right)}$$

and using this form (Abramowitz and Stegun [1]), it can be shown that Hankel function satisfies the far-field radiation condition. Thus $\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\phi_S}{\partial r} - ik\phi_S\right) \to 0$. So Hankel function will consitute a solution to the scattered potential. Combining all relevant solutions, we can write the incident potential as

$$\phi_I = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\sum_{m=0}^{\infty}\left[\epsilon_m i^m J_m(kr)\right]\cos m\theta\,, \tag{24}$$

and the scattered potential as

$$\phi_S = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\sum_{m=0}^{\infty}\left[C_m H_m^{(1)}(kr)\right]\cos m\theta\,, \tag{25}$$

where A is the wave amplitude, $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$. The constants C_m's are to be determined from boundary condition (17). Now the total complex velocity potential becomes

$$\phi(r,\theta,z) = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh} \times \sum_{m=0}^{\infty} \epsilon_m i^m \left[J_m(kr) - \frac{J_m'(kb)}{H_m^{(1)'}(kb)} H_m^{(1)}(kr) \right] \cos m\theta \,. \quad (26)$$

Now the dynamic pressure due to the waves at the surface of the cylinder, $r = b$,

$$p = -\rho \left(\frac{\partial \Phi}{\partial t} \right)_{r=b} . \quad (27)$$

Thus we have

$$p = -\mathrm{Re} \left[\frac{2g\rho A}{\pi kb} \frac{\cosh k(z+h)}{\cosh kh} \sum_{m=0}^{\infty} \frac{\epsilon_m i^m \cos m\theta}{J_m'^2(kb) + Y_m'^2(kb)} \times \left\{ J_m'(kb) \cos \sigma t - Y_m'(kb) \sin \sigma t \right\} - i \left\{ J_m'(kb) \sin \sigma t + Y_m'(kb) \cos \sigma t \right\} \right] . \quad (28)$$

Here we use the following relation

$$J_m(kb) H_m^{(1)'}(kb) - J_m'(kb) H_m^{(1)}(kb) = \frac{2i}{\pi kb} . \quad (29)$$

The above equation can be used to calculate the force in the positive x-direction per unit length using $F_x = \int_0^{2\pi} (-p\, b \cos\theta)\, d\theta$. Since there is no contribution from the $y-$component to the horizontal force as incident wave is parallel to the $x-$axis, the total horizontal force is evaluated using $\int_{-h}^{0} F_x dz$. Since only $\cos\theta$ appears in the integral, the contribution to the force from p will come from the term corresponding to $m = 1$. This yields total horizontal force, F_H, as

$$F_H = \int_{-h}^{0} \int_0^{2\pi} \frac{4g\rho A}{\pi k} \frac{\cosh k(z+h)}{\cosh kh} \frac{J_1'(kb) \sin \sigma t + Y_1'(kb) \cos \sigma t}{J_1'^2(kb) + Y_1'^2(kb)} \cos^2 \theta d\theta dz = \frac{4g\rho A}{k^2} \frac{J_1'(kb) \sin \sigma t + Y_1'(kb) \cos \sigma t}{J_1'^2(kb) + Y_1'^2(kb)} \tanh kh \quad (30)$$

4. Solution for an Elliptical Cylinder

Here we take the cylindrical structure to be elliptical. Considering the elliptical coordinates

$$\begin{aligned} x &= \mu \cosh \xi \cos \eta \,, \\ y &= \mu \sinh \xi \sin \eta \,, \\ z &= z \,, \end{aligned} \tag{31}$$

we can write the equation (13) as

$$\frac{1}{\mu^2 \left(\cosh^2 \xi - \cos^2 \eta\right)} \left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right) + \frac{\partial^2 \phi}{\partial z^2} \quad = \quad 0 \,,$$

which is equivalent to

$$\frac{2}{\mu^2 \left(\cosh 2\xi - \cos 2\eta\right)} \left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right) + \frac{\partial^2 \phi}{\partial z^2} \quad = \quad 0 \,. \tag{32}$$

Now we use the method of separation of variables to solve this problem. By writing

$$\phi(\xi, \eta, z) = F(\xi)\, G(\eta)\, Z(z) \,, \tag{33}$$

we get from equation (32)

$$\frac{2}{\mu^2 \left(\cosh 2\xi - \cos 2\eta\right)} \left(\frac{1}{F} \frac{d^2 F}{d\xi^2} + \frac{1}{G} \frac{d^2 G}{d\eta^2} \right) + \frac{1}{Z} \frac{d^2 Z}{dz^2} \quad = \quad 0 \,.$$

Now separating variable z first, we have the following

$$\frac{2}{\mu^2 \left(\cosh 2\xi - \cos 2\eta\right)} \left(\frac{1}{F} \frac{d^2 F}{d\xi^2} + \frac{1}{G} \frac{d^2 G}{d\eta^2} \right) = -\frac{1}{Z} \frac{d^2 Z}{dz^2} \quad = \quad -k^2 \,, \tag{34}$$

where k is the constant of separation. Separating the variables ξ and η, we can write

$$\frac{1}{F} \frac{d^2 F}{d\xi^2} + \frac{\mu^2 k^2}{2} \cosh 2\xi = -\frac{1}{G} \frac{d^2 G}{d\eta^2} + \frac{\mu^2 k^2}{2} \cos 2\eta \quad = \quad a \,, \tag{35}$$

where a is the constant of separation. Equations (34) and (35) yield three ordinary differential equations

$$\begin{aligned} \frac{d^2 Z}{dz^2} - k^2 Z &= 0\,, & (36)\\ \frac{d^2 G}{d\eta^2} + (a - 2q\cos 2\eta)\, G &= 0\,, & (37)\\ \frac{d^2 F}{d\xi^2} - (a - 2q\cosh 2\xi)\, F &= 0\,, & (38) \end{aligned}$$

where $q = \left(\frac{\mu k}{2}\right)^2$.

Equation (37) is called Mathieu equation and equation (38) as modified Mathieu equation. Periodic Mathieu functions of the first kind, $ce_r(\eta, q)$ and $se_r(\eta, q)$, are the solutions of the equation (37). The functions $ce_r(\eta, q)$ and $se_r(\eta, q)$ are called even and odd respectively. If r is even, then corresponding Mathieu functions have period π and if r is odd, then corresponding Mathieu functions have period 2π. Modified Mathieu functions of first kind, $\mathrm{Ce}_r(\xi, q)$ and $\mathrm{Se}_r(\xi, q)$, are the solutions of the equation (38). Notations used here are based on the notations of McLachlan [9]. Also description of Mathieu functions and their properties have been presented by other authors (Abramowitz [1]). Tables relating to various Mathieu functions are also available in literature. Since the incident wave is parallel to the $x-$axis, we are interested in the solution containing the terms involving $ce_r(\eta, q)$ and $Ce_r(\xi, q)$.

We know that the incident velocity potential, $\phi_I(\xi, \eta, z)$, and scattered velocity potential, $\phi_S(\xi, \eta, z)$, contribute towards the total velocity potential, $\phi(\xi, \eta, z)$. Also from equation (36) and the bottom boundary condition, we have

$$Z(z) = \frac{gA}{\sigma} \frac{\cosh k(z+h)}{\cosh kh}\,, \tag{39}$$

where k satisfies the dispersion relation $\sigma^2 = gk \tanh kh$ and A is the wave amplitude. The dispersion relation is obtained from the free surface condition. Now we write

$$\phi(\xi, \eta, z) = \phi_I(\xi, \eta, z) + \phi_S(\xi, \eta, z)\,,$$

or

$$\begin{aligned} \phi(\xi, \eta, z) = Z(z)\psi(\xi, \eta) &= Z(z)\left\{\psi_I(\xi, \eta) + \psi_S(\xi, \eta)\right\}\\ &= \frac{gA}{\sigma} \frac{\cosh k(z+h)}{\cosh kh} \left\{\psi_I(\xi, \eta) + \psi_S(\xi, \eta)\right\}\,, \qquad (40) \end{aligned}$$

where $\psi_I(\xi,\eta)$ and $\psi_S(\xi,\eta)$ are due to incident wave and scattered wave respectively.

Using

$$e^{ikx} = e^{ik\mu\cosh\xi\cos\eta} = \sum_{m=-\infty}^{\infty} i^m J_m(k\mu\cosh\xi)e^{im\eta}\,, \tag{41}$$

and grouping even and odd terms and considering the fact that $k\mu = 2\sqrt{q}$, we get

$$\begin{aligned}
e^{i2\sqrt{q}\cosh\xi\cos\eta} &= \sum_{m=-\infty}^{\infty} i^m J_m(2\sqrt{q}\cosh\xi)e^{im\eta} \\
&= J_0(2\sqrt{q}\cosh\xi) + 2\sum_{s=1}^{\infty}(-1)^s J_{2s}(2\sqrt{q}\cosh\xi)\cos(2s\eta) \\
&\quad + 2i\sum_{s=0}^{\infty}(-1)^s J_{2s+1}(2\sqrt{q}\cosh\xi)\cos[(2s+1)\eta] \\
&= 2\sum_{s=0}^{\infty}(-1)^s J_{2s}(2\sqrt{q}\cosh\xi)\sum_{n=0}^{\infty} A_{2s}^{(2n)} ce_{2n}(\eta,q) \\
&\quad + 2i\sum_{s=0}^{\infty}(-1)^s J_{2s+1}(2\sqrt{q}\cosh\xi)\sum_{n=0}^{\infty} A_{2s+1}^{(2n+1)} ce_{2n+1}(\eta,q) \\
&= 2\sum_{n=0}^{\infty} ce_{2n}(\eta,q)\sum_{s=0}^{\infty} A_{2s}^{(2n)}(-1)^s J_{2s}(2\sqrt{q}\cosh\xi) \\
&\quad + 2i\sum_{n=0}^{\infty} ce_{2n+1}(\eta,q)\sum_{s=0}^{\infty} A_{2s+1}^{(2n+1)}(-1)^s J_{2s+1}(2\sqrt{q}\cosh\xi)\,,
\end{aligned} \tag{42}$$

where $A_r^{(m)}$s are Fourier coefficients of the periodic Mathieu functions. Here we use the results (McLachlan [9], p. 210)

$$\cos 2rz = \sum_{n=0}^{\infty} A_{2r}^{(2n)} ce_{2n}(z,q)\,, \tag{43}$$

$$\cos(2r+1)z = \sum_{n=0}^{\infty} A_{2r+1}^{(2n+1)} ce_{2n+1}(z,q)\,, \tag{44}$$

where

$$2\sum_{n=0}^{\infty}\left[A_0^{(2n)}\right]^2 = 1\,, \tag{45}$$

$$\sum_{n=0,r>0}^{\infty}\left[A_{2r}^{(2n)}\right]^2=\sum_{n=0}^{\infty}\left[A_{2r+1}^{(2n+1)}\right]^2=1\,, \tag{46}$$

and

$$\sum_{n=0}^{\infty}\left[A_{2r}^{(2n)}A_{2s}^{(2n)}\right]=\sum_{n=0}^{\infty}\left[A_{2r+1}^{(2n+1)}A_{2s+1}^{(2n+1)}\right]=0 \tag{47}$$

provided $r \neq s$.

Using the following two relations for the modified Mathieu functions of first kind (Abramowitz and Stegun [7], p. 733, McLachlan [9] pp. 159, 160)

$$Ce_{2n}(\zeta,q)=\frac{ce_{2n}(\frac{\pi}{2},q)}{A_0^{(2n)}}\sum_{s=0}^{\infty}(-1)^s A_{2s}^{(2n)}J_{2s}(2\sqrt{q}\cosh\zeta)\,, \tag{48}$$

$$Ce_{2n+1}(\zeta,q)=\frac{ce'_{2n+1}(\frac{\pi}{2},q)}{\sqrt{q}A_1^{(2n+1)}}\sum_{s=0}^{\infty}(-1)^{s+1}A_{2s+1}^{(2n+1)}J_{2s+1}(2\sqrt{q}\cosh\zeta)\,, \tag{49}$$

$\psi_I(\xi,\eta)$ can be expressed as

$$\psi_I(\xi,\eta)=2\sum_{n=0}^{\infty}\left[\frac{A_0^{(2n)}Ce_{2n}(\xi,q)ce_{2n}(\eta,q)}{ce_{2n}(\frac{\pi}{2},q)}\right.\\ \left.-\frac{i\sqrt{q}A_1^{(2n+1)}Ce_{2n+1}(\xi,q)ce_{2n+1}(\eta,q)}{ce'_{2n+1}(\frac{\pi}{2},q)}\right]\,, \tag{50}$$

where $'$ denotes the differentiation with respect to the argument. This can be also written as

$$\psi_I(\xi,\eta)=\sum_{n=0}^{\infty}\left[\alpha_{2n}Ce_{2n}(\xi,q)ce_{2n}(\eta,q)\right.\\ \left.+i\alpha_{2n+1}Ce_{2n+1}(\xi,q)ce_{2n+1}(\eta,q)\right]\,, \tag{51}$$

where

$$\alpha_{2n}=\frac{2A_0^{(2n)}}{ce_{2n}(\frac{\pi}{2},q)}\,,\quad \alpha_{2n+1}=-\frac{2\sqrt{q}A_1^{(2n+1)}}{ce'_{2n+1}(\frac{\pi}{2},q)}\,. \tag{52}$$

Now we try find an expression for ψ_S which satisfies the governing equations and the various boundary conditions previously outlined. Assuming that the scattered wave takes the similar type of expansion but based on the modified

Mathieu function of third kind (Mathieu-Hankel function), $Me_m^{(1)}(\xi, q)$, with initially unknown coefficients β_m, we have

$$\begin{aligned}\psi_S(\xi,\eta) &= \sum_{n=0}^{\infty}\Big[\beta_{2n} Me_{2n}^{(1)}(\xi,q) ce_{2n}(\eta,q) \\ &\quad + i\beta_{2n+1} Me_{2n+1}^{(1)}(\xi,q) ce_{2n+1}(\eta,q)\Big]\,. \end{aligned} \tag{53}$$

Here

$$Me_m^{(1)}(\xi,q) = Ce_m(\xi,q) + iFey_m(\xi,q)\,, \tag{54}$$

where $Ce_m(\xi,q)$ and $Fey_m(\xi,q)$ are the modified Mathieu functions of first and second kind respectively, and β_m are constants to be determined from boundary conditions.

The modified Mathieu function of third kind satisfies the far field radiation condition as $\xi \to \infty$. Here we have

$$x = \mu\cosh\xi\cos\eta\,, \quad y = \mu\sinh\xi\sin\eta\,, \quad r^2 = x^2 + y^2$$

When $\xi \to \infty$ then $\cosh\xi \simeq \sinh\xi \simeq \frac{e^\xi}{2}$. So we have

$$r = \frac{\mu e^\xi}{2}\,.$$

Hence the far field radiation condition becomes

$$\lim_{\xi\to\infty}\sqrt{\frac{\mu e^\xi}{2}}\left(\frac{\partial\psi_S}{\partial\xi} - ik\psi_S\right) \to 0\,. \tag{55}$$

Assuming the asympototic expansion of $Me_m^{(1)}$ for large argument, it can be shown that $Me_m^{(1)}$ satisfies the far field radiation condition.

Now applying the body surface boundary condition

$$\frac{\partial\psi_S}{\partial\xi} = -\frac{\partial\psi_I}{\partial\xi} \quad \text{at} \quad \xi = \xi_0 \tag{56}$$

we get

$$\beta_{2n} = -\frac{Ce'_{2n}(\xi_0,q)}{Me_{2n}^{(1)'}(\xi_0,q)}\alpha_{2n}\,, \tag{57}$$

$$\beta_{2n+1} = -\frac{Ce'_{2n+1}(\xi_0,q)}{Me_{2n+1}^{(1)'}(\xi_0,q)}\alpha_{2n+1}\,. \tag{58}$$

So ψ_S becomes

$$\psi_S(\xi,\eta) = \sum_{n=0}^{\infty}\left[\alpha_{2n}\left\{-\frac{Ce'_{2n}(\xi_0,q)}{Me^{(1)'}_{2n}(\xi_0,q)}Me^{(1)}_{2n}(\xi,q)\right\}ce_{2n}(\eta,q)\right.$$
$$\left. + i\alpha_{2n+1}\left\{-\frac{Ce'_{2n+1}(\xi_0,q)}{Me^{(1)'}_{2n+1}(\xi_0,q)}Me^{(1)}_{2n+1}(\xi,q)\right\}ce_{2n+1}(\eta,q)\right]. \quad (59)$$

Now the total velocity potential becomes

$$\phi(\xi,\eta,z) = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\{\psi_I(\xi,\eta)+\psi_S(\xi,\eta)\} = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}$$
$$\times\sum_{n=0}^{\infty}\left[\alpha_{2n}\left\{Ce_{2n}(\xi,q)-\frac{Ce'_{2n}(\xi_0,q)}{Me^{(1)'}_{2n}(\xi_0,q)}Me^{(1)}_{2n}(\xi,q)\right\}ce_{2n}(\eta,q)\right.$$
$$\left. + i\alpha_{2n+1}\left\{Ce_{2n+1}(\xi,q)-\frac{Ce'_{2n+1}(\xi_0,q)}{Me^{(1)'}_{2n+1}(\xi_0,q)}Me^{(1)}_{2n+1}(\xi,q)\right\}ce_{2n+1}(\eta,q)\right]. \quad (60)$$

Since α_{2n} and α_{2n+1} are known, the velocity potential due to the diffraction is completely known and it can be used to compute wave forces on the structure.

The respective wave force components F_x and F_y in the positive x-direction and in the positive y-direction are evaluated from the pressure distribution $p(\xi_0,\eta,z,t) = -\rho\left(\frac{\partial\Phi}{\partial t}\right)_{\xi=\xi_0}$, where

$$\Phi(\xi,\eta,z,t) = Re\left[\phi(\xi,\eta,z)\,e^{-i\sigma t}\right].$$

Thus the force components are given by

$$F_x = \int_{-h}^{0}\int_{0}^{2\pi} -p(\xi_0,\eta,z,t)\mu\sinh\xi_0\cos\eta d\eta dz\,, \quad (61)$$
$$F_y = \int_{-h}^{0}\int_{0}^{2\pi} -p(\xi_0,\eta,z,t)\mu\cosh\xi_0\sin\eta d\eta dz\,. \quad (62)$$

In this particular case, the $y-$component of the force is zero since the incident wave is parallel to $x-$axis.

5. Limiting Case: From Elliptical Cylinder

to Circular Cylinder

From the elliptical coordinates $x = \mu\cosh\xi\cos\eta$ and $y = \mu\sinh\xi\sin\eta$, we have the following equations

$$\frac{x^2}{\mu^2\cosh^2\xi} + \frac{y^2}{\mu^2\sinh^2\xi} = 1\,, \tag{63}$$

$$\frac{x^2}{\mu^2\cos^2\eta} - \frac{y^2}{\mu^2\sin^2\eta} = 1\,. \tag{64}$$

The equation (63) represents a family of confocal ellipses with major axes $2\mu\cosh\xi$, minor axes $2\mu\sinh\xi$, the common foci being the points $x = \pm\mu$, $y = 0$. The equation (64) represents a family of confocal hyperbolas with the same foci. The two families intersect orthogonally at the points defined by the coordinates $x = \mu\cosh\xi\cos\eta$, $y = \mu\sinh\xi\sin\eta$. At the extremities of the major axis of any confocal ellipse, $\eta = 0,\ \pi$, $y = 0$, and $x = \pm\mu\cosh\xi = \pm r$. For the minor axis, $\eta = \pi/2,\ 3\pi/2$, and $y = \pm\mu\sinh\xi$. The eccentricity of the ellipse, E, is given by $E = \mu/r$. Thus we have $\cosh\xi = 1/E$. When the ellipse tends to a circle of radius r, we have $E = 0$. So we get $\mu \to 0$ and $\xi \to \infty$. The foci tend to coalesce at the origin and $\mu\cosh\xi \to \mu\sinh\xi \to r$.

The equation (64) can be written as

$$\frac{x^2}{\cos^2\eta} - \frac{y^2}{\sin^2\eta} \quad = \quad \mu^2\,. \tag{65}$$

When $\mu \to 0$, $y/x \to \pm\tan\eta$, we have $\eta \to \theta$, $\cos\eta \to \cos\theta$. The confocal hyperbolas become radii of the circle and make angles θ with the $x-$axis.

Thus as eccentricity $E \to 0$, $q \to 0$

$$ce_m(\eta, q) \to \cos m\eta = \cos m\theta\,, \tag{66}$$

the confocal hyperbolas become radii of the circle with $\eta = \theta$ and $m \geq 1$. For $m = 0$, $q \to 0$, $ce_0(\eta, 0) \to A_0^{(0)} = \frac{1}{\sqrt{2}}$. When a confocal ellipse of semi-major axis r tends to a circle with this radius, $\xi \to \infty$, $\mu \to 0$ so that $\mu\cosh\xi \to r$, which implies $\frac{\mu e^\xi}{2} \to r$. Thus $2q\cosh 2\xi \to q\,e^{2\xi}$ keeping this finite, $q \to 0$, we get $a \to m^2$. Thus the equation (38) becomes

$$\frac{d^2F}{d\xi^2} + \left(qe^{2\xi} - m^2\right)F \quad = \quad 0\,. \tag{67}$$

Equation (67) can be transformed by putting $r = \frac{\mu e^\xi}{2}$ to

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} + \left(qe^{2\xi} - m^2\right)F \quad = \quad 0\,,$$

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} + \left(\frac{4qr^2}{\mu^2} - m^2\right)F \quad = \quad 0\,,$$

which is equivalent to

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} + \left(k^2r^2 - m^2\right)F \quad = \quad 0\,, \tag{68}$$

since $q = \left(\frac{\mu k}{2}\right)^2$. Equation (68) is the standard Bessel equation of order m with argument kr. Also the equation (37) becomes (as $q \to 0$)

$$\frac{d^2G}{d\theta^2} + m^2G \quad = \quad 0\,.$$

Thus the equations (36)-(38) reduces to equations (19)-(21). So, in the limit, the solution obtained by solving these equations is same as the solution for the circular diffraction problem.

Another way to derive the velocity potential for circular case from the elliptical case, we proceed as follows. We use the following relations as $\xi \to \infty$

$$Ce_{2n}(\xi,q) \to (-1)^n ce_{2n}(0,q)\, ce_{2n}\left(\frac{\pi}{2},q\right) J_{2n}(kr)/A_0^{(2n)}\,, \tag{69}$$

$$Ce_{2n+1}(\xi,q) \to (-1)^{n+1} ce_{2n+1}(0,q)\, ce'_{2n+1} \times \left(\frac{\pi}{2},q\right) J_{2n+1}(kr)/\sqrt{q}A_1^{(2n+1)}\,, \tag{70}$$

$$Me_{2n}^{(1)}(\xi,q) \to (-1)^n ce_{2n}(0,q)\, ce_{2n}\left(\frac{\pi}{2},q\right) H_{2n}^{(1)}(kr)/A_0^{(2n)}\,, \tag{71}$$

$$Me_{2n+1}^{(1)}(\xi,q) \to (-1)^{n+1} ce_{2n+1}(0,q)\, ce'_{2n+1} \times \left(\frac{\pi}{2},q\right) H_{2n+1}^{(1)}(kr)/\sqrt{q}A_1^{(2n+1)}\,, \tag{72}$$

to derive the velocity potential for circular cylinder from elliptical cylinder. Since the $z-$component is same for both potentials, we need to consider

$$\psi(\xi,\eta) = \sum_{n=0}^{\infty}\left[\alpha_{2n}\left\{Ce_{2n}(\xi,q) - \frac{Ce'_{2n}(\xi_0,q)}{Me_{2n}^{(1)'}(\xi_0,q)}Me_{2n}^{(1)}(\xi,q)\right\}ce_{2n}(\eta,q)\right.$$
$$\left. + \, i\alpha_{2n+1}\left\{Ce_{2n+1}(\xi,q) - \frac{Ce'_{2n+1}(\xi_0,q)}{Me_{2n+1}^{(1)'}(\xi_0,q)}Me_{2n+1}^{(1)}(\xi,q)\right\}ce_{2n+1}(\eta,q)\right]\,. \tag{73}$$

Considering the first term in the summation, we can write, for $n \geq 1$

$$\begin{aligned}
&\alpha_{2n} Ce_{2n}(\xi,q)\, ce_{2n}(\eta,q) \\
&\qquad \to \frac{(-1)^n 2A_0^{(2n)} ce_{2n}(0,q)\, ce_{2n}(\pi/2,q)\, J_{2n}(kr) ce_{2n}(\eta,q)}{A_0^{(2n)} ce_{2n}(\pi/2,q)} \\
&\qquad = 2(-1)^n ce_{2n}(0,q)\, ce(\eta,q)\, J_{2n}(kr) \simeq 2(-1)^n J_{2n}(kr) \cos 2n\theta\,, \quad (74)
\end{aligned}$$

as $ce_m(\eta,q) \to \cos m\eta = \cos m\theta$. Similarly the second, third and fourth terms in the summation ($n \geq 1$) in equation (73) respectively can be written as

$$\alpha_{2n} \frac{Ce'_{2n}(\xi_0,q)}{Me_{2n}^{(1)'}(\xi_0,q)} Me_{2n}^{(1)}(\xi,q)\, ce_{2n}(\eta,q) \to 2(-1)^n \frac{J'_{2n}(kb)}{H_{2n}^{(1)'}(kb)} H_{2n}^{(1)}(kr) \cos 2n\theta\,, \quad (75)$$

$$i\alpha_{2n+1} Ce_{2n+1}(\xi,q)\, ce_{2n+1}(\eta,q) \to 2i(-1)^n J_{2n+1}(kr) \cos(2n+1)\theta\,, \quad (76)$$

$$i\alpha_{2n+1} \frac{Ce'_{2n+1}(\xi_0,q)}{Me_{2n+1}^{(1)'}(\xi_0,q)} Me_{2n+1}^{(1)}(\xi,q)\, ce_{2n+1}(\eta,q) \to 2i(-1)^n \frac{J'_{2n+1}(kb)}{H_{2n+1}^{(1)'}(kb)} H_{2n+1}^{(1)}(kr) \cos(2n+1)\theta\,. \quad (77)$$

When $n = 0$, $q \to 0$, we have $ce_0(\eta,0) \to A_0^{(0)} = \frac{1}{\sqrt{2}}$. So corresponding $n = 0$, we have

$$\alpha_0 Ce_0(\xi,q)\, ce_0(\eta,q) \to 2\left[A_0^{(0)}\right]^2 J_0(kr) = 2\left[\frac{1}{\sqrt{2}}\right]^2 J_0(kr) = J_0(kr)\,, \quad (78)$$

and

$$\alpha_0 \frac{Ce'_0(\xi_0,q)}{Me_0^{(1)'}(\xi_0,q)} Me_0^{(1)}(\xi,q)\, ce_0(\eta,q) \quad \to \quad \frac{J'_0(kb)}{H_0^{(1)'}(kb)} H_0^{(1)}(kr)\,. \quad (79)$$

Thus in the limit we have

$$\psi(r,\theta) = \sum_{n=0}^{\infty} (-1)^n \left[\epsilon_{2n} \left\{ J_{2n}(kr) - \frac{J'_{2n}(kb)}{H_{2n}^{(1)'}(kb)} H_{2n}^{(1)}(kr) \right\} \cos 2n\theta \right.$$

$$+i\epsilon_{2n+1}\left\{J_{2n+1}(kr)-\frac{J'_{2n+1}(kb)}{H^{(1)'}_{2n+1}(kb)}H^{(1)}_{2n+1}(kr)\right\}\cos(2n+1)\theta\Bigg],$$

$$\psi(r,\theta)=\sum_{m=0}^{\infty}\epsilon_m i^m\left\{J_m(kr)-\frac{J'_m(kb)}{H^{(1)'}_m(kb)}H^{(1)}_m(kr)\right\}\cos m\theta\,. \qquad (80)$$

Finally we can write the limiting velocity potential ϕ as

$$\phi=\frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh} \times\sum_{m=0}^{\infty}\epsilon_m i^m\left\{J_m(kr)-\frac{J'_m(kb)}{H^{(1)'}_m(kb)}H^{(1)}_m(kr)\right\}\cos m\theta \qquad (81)$$

which is exactly same as the volicity potential function for the circular cylinder case as derived in Section 3 in equation (26).

6. Conclusion

In this paper, we have considered the water wave diffraction problem due to vertical, surface piercing cylindrical structures. We have solved these two diffraction problems by using the method of separation of variables. Analytical solution for the circular cylinder has been obtained in terms of Bessel and Hankel functions. To solve the diffraction problem for an elliptical cylinder, we have used elliptical coordinates. The analytical solution contains terms involving Mathieu and modified Mathieu functions. Also we have discussed how to obtain the analytical expressions for the wave forces on those structures. As a limiting case, we have derived the velocity potential function of circular cylinder from the velocity potential function of elliptical cylinder by two seperate methods. One way to achieve this, first we have converted the governing Mathieu equation to a Bessel equation as the limiting case, and then the solution has been obtained in terms of Bessel function and Hankel function. For the other method we have used the degenerate form of the Mathieu functions and modified Mathieu functions.

Acknowledgements

The author would like to thank Professor Lokenath Debnath, Chair, Department of Mathematics, Univeristy of Texas-Pan American, Edinburg, Texas, U.S.A., for his encouragement and constructive suggestions during the preparation of this work.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functuions,* Dover Publications, New York (1972).

[2] D.D. Bhatta, M. Rahman, On scattering and radiation problem for a cylinder in water of finite depth, *International Journal of Engineering Science*, **41** (2003), 931-967.

[3] F.P. Chau, R. Eatock Taylor, Second order wave diffraction by a vertical cylinder, *Journal of Fluid Mechanics*, **240** (1992), 571-599.

[4] R.G. Dean, R.A. Dalrymple, *Water Wave Mechanics for Engineers and Scientists*, Prentice-Hall Inc, New Jersey (1984).

[5] L. Debnath, *Nonlinear Water Waves*, Academic Press (1994).

[6] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser, Boston (1997).

[7] C.J.R. Garrett, Wave forces on a circular dock, *Journal of Fluid Mechanics*, **46** (1971), 129-139.

[8] R.C. MacCamy, R.A. Fuchs, *Wave Forces on Piles: A Diffraction Theory*, Technical Memorandum, No. **69**, U.S. Army Corps of Engineers, Beach Erosion Board, Washington, D.C. (1954).

[9] N.W. McLachlan, *Theory and Application of Mathieu Functions,* Oxford University Press (1947).

[10] M. Rahman, *Water Waves: Relating Modern Theory to Advanced Engineering Practice*, Oxford University Press, Cambridge (1994).

[11] M. Rahman, D.D. Bhatta, Evaluation of added mass and damping coefficients of an osciallating circular cylinder, *Applied Mathematical Modelling*, **17** (1993),70-79.