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ESTIMATES OF WEIGHTED EIGENVALUES FOR THE ELASTIC VIBRATING PLATE

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Abstract: In this paper, the corresponding eigenvalue problem for the vibration of elastic plates under the natural boundary condition is considered. The aim of this paper is to give an upper estimate for weighted eigenvalues of the vibrating plate.

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1. Introduction

For Dirichlet eigenvalues of the Laplacian, P. Li and S.T. Yau [4] provided a lower estimate. By means of the analogy (cf. [6]) between the eigenvalue problems under the Dirichlet and Neumann boundary condition respectively, P. Kröger [2], [3] gained an upper bound and a lower bound for the Neumann eigenvalue of the Laplace operator.

On the weighted eigenvalue problems of elliptic operators, X.T. Liu and Z.C. Chen, [5] gave an upper bound for Dirichlet eigenvalues of the Laplacian, but the method is different from ones in the above papers.

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In this paper, the corresponding eigenvalue problem for the vibration of elastic plates under the natural boundary condition is considered. We adapt the technique used by Li, Yau and Kröger for obtaining bounds of eigenvalues of the Laplacian in order to get the upper estimates for weighted eigenvalues of vibrating plates.

Let Ω be a bounded domain in R^m $(m = 2)$ with sufficiently smooth boundary $\partial\Omega$, the following weighted eigenvalue problem with respect to the vibration of elastic plates is considered:

$$\begin{cases} \triangle^2 u(x) = \lambda\rho(x)u(x), & \text{in } \Omega, \\ \triangle u(x) = \dfrac{\partial}{\partial n} \triangle u(x) = 0, & \text{on } \partial\Omega \end{cases} \tag{$*$}$$

in which n is the unit outward normal to $\partial\Omega, \rho(x) \in C(\overline{\Omega})$ is the mass per unit area, and satisfies $0 < \rho(x) \leq \alpha^{-1}$, where α is a constant.

As we know, the problem $(*)$ is the mathematical model for the vibration of elastic plates [1]. Variational methods show that the discreteness of the spectrum of the problem $(*)$ allows us to order the eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, monotonically. If we set $\lambda = \omega^2$, then ω denotes the radian frequency of the vibrating plate.

We consider the general case for $m \geq 2$, our results are the following.

Theorem. *Let Ω be a bounded domain with sufficiently smooth boundary in $R^m (m \geq 2)$. Suppose that $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k+1}$ are the first $k+1$ eigenvalues for the Neumann boundary value problem $(*)$. Then*

$$\lambda_{k+1} \leq \inf_{r > 2\pi(mk/\alpha A(S_1)\int_\Omega \rho(y)dy)^{1/m}} \frac{\alpha A(S_1)V(\Omega)r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^k \lambda_j}{(\alpha A(S_1)r^m/m)\int_\Omega \rho(y)dy - (2\pi)^m k}, \tag{1}$$

where $A(S_1)$ denotes the area of the $(m-1)$-unit sphere in R^m, $V(\Omega)$ is the volume of the domain Ω.

Corollary 1. *Under the assumptions of the theorem, the following inequality holds:*

$$\sum_{j=1}^{k+1} \lambda_j \leq \frac{m}{m+4}(2\pi)^4 (k+1)^{(m+4)/m} (m/\alpha A(S_1))^{4/m} \left(\int_\Omega \rho(y)dy \right)^{-4/m}.$$

Proof. The assertion following immediately from the theorem if we set

$$r = 2\pi \left(\alpha A(S_1) \int_\Omega \rho(y)dy \Big/ m(k+1) \right)^{-1/m} . \qquad \square$$

Corollary 2. *Under the assumptions of the theorem, the following inequality holds:*

$$\lambda_{k+1} \leq (2\pi)^4 k^{4/m} (m(m+4)/4\alpha A(S_1))^{4/m} \left(\int_\Omega \rho(y)dy \right)^{-4/m} .$$

Proof. The assertion following immediately from the theorem if we estimate $\sum_{j=1}^m \lambda_j$ below by 0 and if we set

$$r = 2\pi \left(4\alpha A(S_1) \int_\Omega \rho(y)dy \Big/ m(m+4)k \right)^{-1/m} . \qquad \square$$

Proof of Theorem. Let $\{u_j\}_{j=1}^k$ be a set of eigenfunctions for the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, and satisfy

$$\int_\Omega \rho(y) u_i(x) u_j(x) dx = \delta_{ij} .$$

Let $U(x,y) = \sum_{j=1}^k u_j(x)u_j(y)$ for $x, y \in \Omega$, $h_z(y) = e^{iy\cdot z}$, where $y \cdot z = \sum_{i=1}^m y_i z_i$. The following transforms of $u_j(x)$ and $U(x,y)$ with respect to the x-variable is defined by

$$\widehat{u}_j(z) = (2\pi)^{-m/2} \int_\Omega \rho(x) u_j(x) e^{ix\cdot z} dx, \quad \widehat{U}(z,y) = \sum_{j=1}^k \widehat{u}_j(z) u_j(y) .$$

Then, the projection of $h_z(y)$ onto the subspace $\mathrm{Span}\{u_1, u_2, \cdot, u_k\}$ of $L^2_\rho(\Omega)$ can be written in terms of the above transform $\widehat{U}$ of U with respect to the x-variable:

$$\sum_{j=1}^k u_j(y) \int_\Omega \rho(x) h_z(x) u_j(x) dx = \sum_{j=1}^k \widehat{u}_j(z) u_j(y) (2\pi)^{m/2} = (2\pi)^{m/2} \widehat{U}(z,y).$$

Thus for any $z \in R^m$, $h_z(y) - (2\pi)^{m/2}\widehat{U}(z,y)$ is orthogonal to $u_1, u_2, \cdots, u_k$ in the $L^2_\rho(D)$ sense. According to the variational principle, we have

$$\lambda_{k+1} \leq \frac{\int_\Omega \left| \triangle_y (h_z(y) - (2\pi)^{m/2} \widehat{U}(z,y)) \right|^2 dy}{\int_\Omega \rho(y) \left| h_z(y) - (2\pi)^{m/2} \widehat{U}(z,y) \right|^2 dy} .$$

Hence, for arbitrary B_r, we obtain

$$\lambda_{k+1} \le \frac{\int_{B_r}\int_\Omega \left|\triangle_y(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\right|^2 dydz}{\int_{B_r}\int_\Omega \rho(y)\left|h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y)\right|^2 dydz}, \tag{2}$$

where B_r stands for the ball with radius r and center 0 in R^m for an arbitrary r with $r > 2\pi\left(mk/\alpha A(S_1)\int_\Omega \rho(y)dy\right)^{1/m}$.

Now, we calculate the numerator in this formula. Because of $|a-b|^2 = |a|^2 - 2\mathrm{Re}(a-b)\overline{b} - |b|^2$, we have

$$\begin{aligned}
&\int_{B_r}\int_\Omega \left|\triangle_y(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\right|^2 dydz \\
&\quad = \int_{B_r}\int_\Omega |\triangle_y(h_z(y)|^2\, dydz - (2\pi)^m \int_{B_r}\int_\Omega \left|\triangle_y\widehat{U}(z,y)\right|^2 dydz \\
&\quad - 2\mathrm{Re}\int_{B_r}\int_\Omega \triangle_y(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot(2\pi)^{m/2}\overline{\triangle_y\widehat{U}(z,y)}dydz\,.
\end{aligned} \tag{3}$$

The first term on the right-hand side of the above equation (3) can easily be calculated as follows:

$$\int_{B_r}\int_\Omega |\triangle_y(h_z(y)|^2\, dydz = \int_{B_r}\int_\Omega |z|^4 dydz = \frac{r^{m+4}}{m+4}A(S_1)V(\Omega)\,.$$

By Green's second formula and the fact that $u_j(y)$ satisfy the boundary condition of the problem $(*)$, we then have

$$\begin{aligned}
(2\pi)^m \int_{B_r}\int_\Omega \left|\triangle_y\widehat{U}(z,y)\right|^2 dydz &= (2\pi)^m \int_{B_r}\int_\Omega \triangle_y^2\widehat{U}(z,y)\cdot\overline{\widehat{U}(z,y)}dydz \\
&\quad + (2\pi)^m \int_{B_r}\int_{\partial\Omega} \triangle_y\widehat{U}(z,y)\cdot\overline{\frac{\partial}{\partial n_y}\widehat{U}(z,y)}dsdz \\
&\quad - (2\pi)^m \int_{B_r}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\triangle_y\,\widehat{U}(z,y)\cdot\overline{\widehat{U}(z,y)}dsdz \\
&= (2\pi)^m \int_{B_r}\int_\Omega \triangle_y^2\widehat{U}(z,y)\cdot\overline{\widehat{U}(z,y)}dydz = (2\pi)^m\sum_{j=1}^k \lambda_j \int_{B_r} |\widehat{u}_j(z)|^2\, dz\,.
\end{aligned}$$

Notice that $h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y)$ is orthogonal to $u_1, u_2, \cdots, u_k$ in the

$L^2_\rho(D)$ sense, the third term on the right-hand side vanishes. In fact,

$$\begin{aligned}
&\int_{B_r}\int_\Omega \triangle_y(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot(2\pi)^{m/2}\overline{\triangle_y\widehat{U}(z,y)}dydz \\
&\qquad=(2\pi)^{m/2}\int_{B_r}\int_\Omega(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot\overline{\triangle_y^2\widehat{U}(z,y)}dydz \\
&\qquad+(2\pi)^{m/2}\int_{B_r}\int_{\partial\Omega}\frac{\partial}{\partial n_y}(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot\overline{\triangle_y\widehat{U}(z,y)}dsdz \\
&\qquad-(2\pi)^{m/2}\int_{B_r}\int_{\partial\Omega}(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot\overline{\frac{\partial}{\partial n_y}\triangle_y\widehat{U}(z,y)}dsdz \\
&=(2\pi)^{m/2}\sum_{j=1}^k\int_{B_r}\int_\Omega\lambda_j\overline{\widehat{u}_j(z)}\rho(y)(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot u_j(y)dydz=0\,.
\end{aligned}$$

Thus, (3) becomes

$$\begin{aligned}
&\int_{B_r}\int_\Omega\left|\triangle_y(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\right|^2dydz \\
&\qquad=A(S_1)V(\Omega)r^{m+4}/(m+4)-(2\pi)^m\sum_{j=1}^k\lambda_j\int_{B_r}|\widehat{u}_j(z)|^2dz\,.
\end{aligned}\tag{3a}$$

On the other hand,

$$\begin{aligned}
&\int_{B_r}\int_\Omega\rho(y)\left|h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y)\right|^2dydz \\
&\qquad=\int_{B_r}\int_\Omega\rho(y)\left|h_z(y)\right|^2dydz-(2\pi)^m\int_{B_r}\int_\Omega\rho(y)\left|\widehat{U}(z,y)\right|^2dydz \\
&\qquad-2\mathrm{Re}\int_{B_r}\int_\Omega\rho(y)(h_z(y)-(2\pi)^{m/2}\widehat{U}(z,y))\cdot(2\pi)^{m/2}\overline{\widehat{U}(z,y)}dydz \\
&\qquad\qquad=(A(S_1)r^m/m)\int_\Omega\rho(y)dy-(2\pi)^m\sum_{j=1}^k\int_{B_r}|\widehat{u}_j(z)|^2dz\,.
\end{aligned}\tag{4}$$

Inserting (3a) and (4) in (2), we obtain the inequality

$$\lambda_{k+1}\le\frac{A(S_1)V(\Omega)r^{m+4}/(m+4)-(2\pi)^m\sum_{j=1}^k\lambda_j\int_{B_r}|\widehat{u}_j(z)|^2dz}{(A(S_1)r^m/m)\int_\Omega\rho(y)dy-(2\pi)^m\sum_{j=1}^k\int_{B_r}|\widehat{u}_j(z)|^2dz}\,.\tag{5}$$

Clearly,

$$\lambda_{k+1} \leq \frac{\alpha A(S_1)V(\Omega)r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^{k} \lambda_j \int_{B_r} \alpha|\widehat{u}_j(z)|^2 dz}{(\alpha A(S_1)r^m/m)\int_\Omega \rho(y)dy - (2\pi)^m \sum_{j=1}^{k} \lambda_j \int_{B_r} \alpha|\widehat{u}_j(z)|^2 dz}\,. \tag{5a}$$

In the following, we adopt the induction method in order to prove the assertion of the theorem in this paper. By Parseval's identity, we know that

$$0 \leq \int_{B_r} \alpha|\widehat{u}_j(z)|^2 dz < \int_{R^m} \alpha|\widehat{u}_j(z)|^2 dz = \int_\Omega \alpha|\rho(y)u_j(y)|^2 dy \leq \int_\Omega \rho(y)u_j^2(y)dy = 1\,. \tag{6}$$

Thus, when $k = 1$, $r > 2\pi \left(m/\alpha A(S_1)\int_\Omega \rho(y)dy\right)^{1/m}$, from (5a), it is clearly seen that

$$\lambda_2 \leq \frac{\alpha A(S_1)V(\Omega)r^{m+4}/(m+4)}{(\alpha A(S_1)r^m/m)\int_\Omega \rho(y)dy - (2\pi)^m}\,. \tag{7}$$

Now, suppose that

$$\lambda_k \leq \frac{\alpha A(S_1)V(\Omega)r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^{k-1} \lambda_j}{(\alpha A(S_1)r^m/m)\int_\Omega \rho(y)dy - (2\pi)^m(k-1)} \tag{8}$$

holds, where $r > 2\pi \left(m(k-1)/\alpha A(S_1)\int_\Omega \rho(y)dy\right)^{1/m}$. Namely,

$$\lambda_k[(\alpha A(S_1)r^m/m)\int_\Omega \rho(y)dy - (2\pi)^m(k-1)] \leq \alpha A(S_1)V(\Omega)r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^{k-1} \lambda_j\,.$$

It follows that

$$\lambda_k \leq \frac{\alpha A(S_1)V(\Omega)r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^{k} \lambda_j}{(\alpha A(S_1)r^m/m)\int_\Omega \rho(y)dy - (2\pi)^m k} \tag{9}$$

holds, when $r > 2\pi \left(mk/\alpha A(S_1)\int_\Omega \rho(y)dy\right)^{1/m}$.

On the other hand, notice that by (6) and the monotony of $\{\lambda_j\}_{j=1}^k$, we have

$$\frac{(2\pi)^m \sum_{j=1}^{k} \lambda_j \left(1 - \int_{B_r} \alpha|\widehat{u}_j(z)|^2 dz\right)}{(2\pi)^m \sum_{j=1}^{k} \left(1 - \int_{B_r} \alpha|\widehat{u}_j(z)|^2 dz\right)} \leq \lambda_k\,. \tag{10}$$

Hence,

$$\frac{(2\pi)^m \sum_{j=1}^{k} \lambda_j \left(1 - \int_{B_r} \alpha |\widehat{u}_j(z)|^2 dz\right)}{(2\pi)^m \sum_{j=1}^{k} \left(1 - \int_{B_r} \alpha |\widehat{u}_j(z)|^2 dz\right)} \leq \frac{\alpha A(S_1) V(\Omega) r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^{k} \lambda_j}{(\alpha A(S_1) r^m/m) \int_\Omega \rho(y) dy - (2\pi)^m k}.$$

Since $\frac{a+c}{b+d} \leq \frac{c}{d}$ holds when $\frac{a}{b} \leq \frac{c}{d}$ $(a, b, c, d > 0)$, from (11) and (5a), it can be seen that

$$\lambda_{k+1} \leq \frac{\alpha A(S_1) V(\Omega) r^{m+4}/(m+4) - (2\pi)^m \sum_{j=1}^{k} \lambda_j}{(\alpha A(S_1) r^m/m) \int_\Omega \rho(y) dy - (2\pi)^m k}.$$

This completes the proof of the theorem. □

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