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ABOUT THE CHARACTERIZATION OF CONVEX KERNEL

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Abstract: Some characterizations of convex kernel of a starshaped set is given by using a version of Krasnoselsky's Separation Lemma given by E.E. Robkin involving points of spherical support and some corollaries are stated.

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1. Introduction

All the points and sets being considered here are assumed to belong to some d dimensional Euclidean space E^d. If x, y are points in some such set S such that $S \subset E^d$, $[x;y]$ denotes the closed line segment whose end points are x and y. The interior, closure, boundary of a set S are respectively denoted by int$S$, cl$S$, bdry$S$. The interior of a set S in a linear space relative to the minimal flat containing it is denoted by rel int$S$. The affine hull of a set S is the intersection of all affine sets which contain S, and it is denoted by aff$S$.

Given any set S and points x and y in this set, the point x sees the point y via S iff the closed line segment $[x;y]$ entirely contained in S (i.e. $[x;y] \subset S$)

Given any set S, the convex hull of S, denoted by conv$S$ is the smallest convex set containing S ; conv$S$ can be defined as the intersection of all the convex sets M containing S; conv$S = \bigcap_{M \supset S} M$.

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Given any set S and a point x contained in this set, the star of x, denoted by st(x, S) is the set of all points of S that see x via S, hence $st(x, S) = \{y \in S : y$ sees x via $S\}$.

Given any set S, the convex kernel of S, denoted by ker S, is defined as the set $\ker S = \{x \in S : st(x, S) = S\}$.

The set S, is convex if ker $S = S$; the set S is starshaped if ker $S \neq \emptyset$, i.e. if there exists at least one point x in S that sees every other point of the set. Consequently a non void convex set is starshaped.

Characterization of starshapedness is one of the main research topics of convex geometry. We consider two problem from this topic, one is describing the convex kernel of a starshaped set S as the intersection of a certain family of subsets of S the other one is finding necessary and sufficient conditions that the convex kernel of S have dimension α, where $0 \leq \alpha \leq d$ and d is the space dimension. For a detailed overview on the subject the reader is referred to the survey article of Fausto Toranzos [4].

In 1946 Krasnoselsky proved that a compact set S in E^d is starshaped if and only if for each subset of $d + 1$ points of S there exists a point of S that can see via S all these points. The proof of this statement use a preliminary fact called Krasnoselsky's Lemma depending to the following concept:

A point $x \in S$ is a regular point of S if there exists a closed halfspace which contain $st(x, S)$ and which has x in its bounding hyperplane.

Lemma 1. (Krasnoselsky [5]) *Suppose S is a closed set in E^d. If $y \in S$, $x \in E^d$ with $[x; y] \nsubseteq S$, then there exist a regular point $z \in S$ and a hyperplane $H = \langle f : \alpha \rangle$ through z which separates the $st(z, S)$ from x, so that $f(st(z, S)) \geq \alpha$, $f(x) < \alpha$.*

This lemma enable us to deduce a formula of convex kernel via stars of regular points. First published result of this fact has given by L. Bragard [1]. E.E. Robkin gave another characterization of starshapedness, in the same spirit of Krasnoselsky, by using point of spherical support. In this paper we give some formulas and results for convex kernel, using a modification of Krasnoselsky's Lemma with point of spherical support.

Definition 2. (see [3]) A non-degenerate closed solid sphere K is said to be a sphere of support to a set S if the closure of S intersects S and if S does not intersect the interior of K. A point x in the intersection of K and cl$S$, is said to be *a point of spherical support of* S. The set S is said to be *spherically supported at* x by K. We will denote the set of points of spherical support of a set S by sph$S$.

Note that a point x of cl$S$ may belong to more than one sphere of support to S and that a sphere K may support S at more than one point.

E.E. Robkin modified Krasnoselsky's Lemma by using point of spherical support. In that case we reach the same conclusion. It is well however, to point out that a regular point of a set need not be a point of spherical support so that the new hypotheses will be weaker than the original. There is a simple example which will demonstrate this.

Example 3. (see [3]) In E^2 take a set of rays from the origin lying in the first quadrant and converging to the positive x axis and a set of rays from the origin lying in the fourth quadrant and also converging to the positive x axis. Then any point on the positive x axis is a regular point of S but not a point of spherical support to S.

Lemma 4. (see [3]) *Let S be a closed set in E^d. Suppose that there is a point u in E^d and a point x in S such that the segment $[u;x]$ is not contained in S. Then there exists a point z of spherical support of S and a sphere K of support to S at z such that the closed halfspace $H(z)$ of support to K at z contains the z-star $st(z,S)$ and not u.*

Notation 5. Consider a set S and $x \in \mathrm{sph} S$. The family of closed halfspaces supporting spheres of support at the point of x will be denoted by $\{H_i(x) : i \in A_x\}$ and the intersection of these closed halfspaces by $C_x = \cap\{H_i(x) : i \in A_x\}$.

Theorem 6. *Let S be a closed set in E^d. Then the convex kernel of S is equal to the intersection of closed support halfspaces supporting spheres of support at the points of spherical support, by the above notation* $\ker S = \bigcap\limits_{y \in \mathrm{sph}\, S} C_y$.

Proof. Let $x \in \ker S$ but $x \notin \bigcap\limits_{y \in \mathrm{sph} S} C_y$. Then there exists at least one $v \in \mathrm{sph} S$ such that for the family $\{H_i(v) : i \in A_v\}$ there exist at least one index $j \in A_v$ with $x \notin H_j(v)$. That means $\mathrm{st}(v,S) \nsubseteq H_j(v)$. But $x \in \ker S$ gives $[x;v] \subset S$, contradiction. So $\ker S \subset \bigcap\limits_{y \in \mathrm{sph} S} C_y$.

Suppose $x \in \bigcap\limits_{y \in \mathrm{sph} S} C_y$ but $x \notin \ker S$. In that case there exists some $b \in S$ such that $[x;b] \nsubseteq S$. Then from the Lemma 4 there exists some point of spherical support z and a hyperplane H such that $z \in H$, $\mathrm{st}(z,S) \subset H(z)$ and H separates x and $\mathrm{st}(z,S)$. But this contradicts with $x \in \bigcap\limits_{y \in \mathrm{sph} S} C_y$. □

Theorem 7. *Let S be a closed set in E^d. Then*

$$\ker S = \bigcap_{y \in \mathrm{sph}S} \mathrm{st}(y, S).$$

Proof. Take any $u \in \mathrm{sph}S$ then closed halfspaces supporting support spheres of u at the point of u contains $\mathrm{st}(u, S)$ that is $\mathrm{st}(u, S) \subset \cap\{H_i(u) : i \in A_u\} = C_u$ then $\bigcap_{y \in \mathrm{sph}S} \mathrm{st}(y, S) \subset \bigcap_{y \in \mathrm{sph}S} C_y = \ker S$.

Assume that $a \in \ker S$ but $a \notin \bigcap_{x \in \mathrm{sph}S} \mathrm{st}(x, S)$. Then there exists at least one $y \in \mathrm{sph}S$ such that $a \notin \mathrm{st}(y, S)$. But $a \in \ker S$ gives $[a; y] \subset S$ contradicts with $a \notin \mathrm{st}(y, S)$. □

Corollary 8. *A closed set E^d is starshaped if and only if*

$$\bigcap_{x \in \mathrm{sph}S} st(x, S) \neq \emptyset.$$

Corollary 9. *Let S be a set in E^d different from the E^d and $\emptyset$. Then S has at least one point of spherical support.*

Corollary 10. *Let S be a closed, nonconvex set in E^d, and T' be the set of points of spherical support of S not containing any point of* $\ker S$. *Then* $\ker S = \bigcap_{x \in T'} \mathrm{st}(x, S)$.

Proof. If $x \in \ker S$ then $\mathrm{st}(x, S) = S$ so $\ker S = \left[\bigcap_{x \in T'} \mathrm{st}(x, S)\right] \cap S = \bigcap_{x \in T'} \mathrm{st}(x, S)$. □

Once we have obtained the formula $\ker S = \bigcap_{x \in \mathrm{sph}S} \mathrm{st}(x, S)$ using a theorem format in [4] we can say the following theorem.

Theorem 11. *Let S be a closed starshaped set in E^d.* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an $\alpha-$dimensional flat F, a point $x \in$*relint$(F \cap S)$ *and a neighborhood U_x of x such that $\forall t \in \mathrm{sph}S$, $U_x \cap F \cap S \subset \mathrm{st}(t, S)$.*

Proof. Assume that there exists an $\alpha-$dimensional flat F, a point $x \in \mathrm{relint}(F \cap S)$ and a neighborhood U_x of x such that $\forall t \in \mathrm{sph}S$, $U_x \cap F \cap S \subset \mathrm{st}(t, S)$ then $\dim(U_x \cap F \cap S) = \alpha$ and the inclusion $\ker S \subset \bigcap_{t \in \mathrm{sph}S} \mathrm{st}(t, S)$ gives $\dim(\ker S) \geq \alpha \geq 0$.

For the converse implication it is enough to take $F = aff \ker S$ and $x \in \mathrm{relint} \ker S$. □

Let $A \subset E^d$ ba a nonempty and closed set. We set, for $x \in E^d \backslash A$, $r_A(x) = inf\{\|x-y\| : y \in A\}$ and denote by $K_A(x)$ the closed ball with center x and radius $r_A(x)$. Every point of $A \cap K_A(x)$ is called a foot x on A. We say that x is skeletal for A, if there is no point $y \neq x$ in $E^d \backslash A$ such that $K_A(x) \subset K_A(y)$. The set of skeletal points for A is denoted by $S(A)$; it contains all points with more than one foot on A. $f \downharpoonleft_X$ is the restriction of the map $f : Y \to Z$ to the subset X of M. Convex deficiency defined as $(\text{conv}A) \setminus A$.

Theorem 12. (Calabi and Hartnet [2]) *Let A and B be closed nonempty subsets of E^d. Then* $\text{conv}(A) \cap (E^d \setminus A)$ $=\text{conv}(B) \cap (E^d \setminus B)$ *if and only if* $\big(S(A), r_A \downharpoonleft_{S(A)}\big) = \big(S(B), r_B \downharpoonleft_{S(B)}\big)$.

Notice that last theorem contains Motzkin characterization of convex sets. Consider the set of points of spherical support determined by skeletal points of a closed set A as $\text{sph}_{S(A)}A$, then we have a reduction of Theorem 7.

Corollary 13. *Let A be a closed set in E^d. Then*

$$\ker A = \bigcap_{x \in \text{sph}_{S(A)}A} \text{st}(x, A).$$

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