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APPROXIMATION OF HOLOMORPHIC MAPS BY ALGEBRAIC MORPHISMS

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Abstract: Let U be an open subset of a smooth complex algebraic curve and Y a smooth projective surface. Here we study the rational approximation of holomorphic maps $f : U \to Y$ keeping fixed one or two values of f following previous work by J. Bochnak and W. Kucharz. We obtain a characterization of the rationality of Y.

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1. Rational Approximation

Our starting point was a nice property of smooth complex projective rational surfaces discovered by J. Bochnak and W. Kucharz ([1], Theorem 1.1 and Corollary 1.2) and their question if this property holds only for rational surfaces (the two lines after [1], Corollary 1.2). We were unable to prove their question and we can only prove the following results which is "morally" much weaker.

Theorem 1. *Let X be a smooth and connected algebraic curve, $P_1, P_2 \in X$ such that $P_1 \neq P_2$ and Y a smooth and connected algebraic surface. Y is rational if and only if for the following property holds: for every $Q \in W$, any open neighborhood U of P in X for the Euclidean topology and any holomorphic map $f : U \to Y$ such that $f(P) = Q$, there is an open neighborhood U' of*

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$\{P_1, P_2\}$ in U and a family $\{h_i\}_{i\in I}$, of rational maps from X onto Y regular on U', with $h_i(P) = f(P)$ for all $i \in I$, and the family $\{h_i\}_{i\in I}$ approximates f on U' for the compact-open topology.

To prove Theorem 1 we will use two easy propositions which may have an independent interest.

Proposition 1. *Let Y be a smooth and connected n-dimensional complex projective variety, X a smooth and connected algebraic curve and $P \in X$. Assume the existence of a non-empty open subset W of Y for the Euclidean topology such that the following property holds: for every $Q \in W$, any open neighborhood U of P in X for the Euclidean topology and any holomorphic map $f : U \to Y$ such that $f(P) = Q$, there is an open neighborhood U' of $\{P_1, P_2\}$ in U and a family $\{h_i\}_{i\in I}$, of rational maps from X onto Y regular on U', with $h_i(P) = f(P)$ for all $i \in I$, and the family $\{h_i\}_{i\in I}$ approximates f on U' for the compact-open topology. Then Y is uniruled.*

Proof. Since every regular map from a Zariski open subset of X into Y extends to a regular map on the unique smooth and connected projective curve $\bar{X}$ such that $X \subseteq \bar{X}$ and $\bar{X}\backslash X$ is finite, extends to a regular map on all $\bar{X}$, without loosing generality we may assume $X = \bar{X}$. We will silently do the same assumption in the proofs of Proposition 1 and Theorem 1. Let $g \geq 0$ be the genus of X. Let $\mathrm{Rat}(Y)$ be the closure in the Hilbert scheme $\mathrm{Hilb}(Y)$ of Y of the irreducible components whose general member is a rational (but perhaps singular) curve ([5], Chapter I and Chapter II, or [3], 5.6); instead of the Hilbert scheme one may use the Chow scheme $\mathrm{Chow}(Y)$ of Y. Since Y is projective, $\mathrm{Hilb}(Y)$ and $\mathrm{Chow}(Y)$ have only countably many connected components and every connected component of $\mathrm{Hilb}(Y)$ or $\mathrm{Chow}(Y)$ is projective ([3], p. 116, or [5], Theorem I.1.4 and Theorem I.3.21) and in particular with the complex topology it is a compact analytic space. For any irreducible component Γ of $\mathrm{Rat}(Y)$ let $J(\Gamma) := \{(P, C) \in Y \times \Gamma : P \in C\}$ be the incidence correspondence. Since Γ is compact, the projection $J(\Gamma) \to Y$ into the first factor is proper and hence its image $I(\Gamma)$ is a closed and irreducible algebraic subset of Y. Since $\mathbb{C}$ is uncountable and $\mathrm{Rat}(Y)$ has only countably many irreducible components, the variety Y is uniruled if and only if there is an irreducible component Γ of $\mathrm{Rat}(Y)$ such that $I(\Gamma) = Y$ ([3], last sentence of Section 1 of Chapter 5). Hence in order to obtain a contradiction we may assume $I(\Gamma) \neq Y$ for all Γ. Set $Y^0 := Y\backslash \bigcup_\Gamma I(\Gamma)$. By Baire Theorem Y^0 is dense in Y for the Euclidean topology. Hence $W \cap Y^0 \neq \emptyset$ for any non-empty open subset W of Y. Fix any such open set W and take any $Q \in W$. If $g = 0$ we obtain (taking any non-constant hoilomorphic map $f : U \to Y$ with $f(P) = Q$ and U small neighborhood of P in X) the existence of a rational curve $C \subseteq Y$ such that $Q \in C$, contradicting the definition of Y^0 and the choice of Q. Now assume $g > 0$. Fix an open neighborhood U of P in X for the Euclidean topology biholomorphic to the unit disk Δ. There is a holomorphic map $f : U \to Y$ such that $f(P) = Q$, f has degree $2g$ onto its image $f(U) \cong \Delta$ and it is totally ramified at P, i.e. (up to the identification of the pairs (U, P) and $(f(U), Q)$ with $(\Delta, 0)$) it is of the form $z \mapsto z^{2g}$. Take a sufficiently near rational approximation $h : X \to Y$ of f and call C the normalization of the curve $f(X)$. Hence h induces a regular map $\phi : X \to C$ whose total degree of ramification is at least $2g-1$. By the Riemann-Hurwitz formula we obtain that C has genus zero. The curve C contradicts the condition $Q \in Y^0$, concluding the proof. □

Proposition 2. *Let Y be a smooth and connected n-dimensional complex projective variety, X a smooth and connected algebraic curve and $P_1, P_2 \in X$ such that $P_1 \neq P_2$. Assume the existence of non-empty open subset W, W' of Y for the Euclidean topology such that the following property holds: for all $(Q_1, Q_2) \in W \times W'$ such that $Q_1 \neq Q_2$, any open neighborhood U of $\{P_1, P_2\}$ in X for the Euclidean topology and any holomorphic map $f : U \to Y$ such that $f(P_j) = Q_j$, $j = 1, 2$, there is an open neighborhood U' of $\{P_1, P_2\}$ in U and a family $\{h_i\}_{i \in I}$, of rational maps from X onto Y regular on U', with $h_i(P_1) = f(P_1)$ and $h_i(P_2) = f(P_2)$ for all $i \in I$, and the family $\{h_i\}_{i \in I}$ approximates f on U' for the compact-open topology. Then Y is rationally connected in the sense of [3], Chapter 4, or [5], Chapter IV.*

Proof. Use that the set $W \times W' \backslash \Delta_{Y \times Y}$ is dense for the Zariski topology on $Y \times Y$, the compactness of any connected component of $\mathrm{Rat}(Y)$ and the definition of rationally connected variety given in [5], Definition IV.3.2.2. □

Proof of Theorem 1. A smooth and projective complex surface is rational if and only if it is rationally connected ([5], Example IV.3.12.2, or use [3], Corollary 4.18, and the classification of surfaces with Kodaira dimension $-\infty$). Hence Proposition 2 gives the "only if" part. Now assume that Y is rational. Use the existence of a non-empty Zariski open subset A of Y which is biregular to a Zariski open subset B of $\mathbf{P}^2$, that the automorphism group of $\mathbf{P}^2$ is 2-transitive and [1], Corollary 1.2. □

Remark 1. There are many smooth and connected projective surfaces Y such that $\mathrm{Rat}(Y)$ is finite. For instance, by a theorem of Bogomolov ([2] or [4], p. 288) this is the case if the minimal model Y' of Y is a surface of general type with $c_1^2(Y') > c_2(Y')$. For any surface Y such that $\mathrm{Rat}(Y)$ is finite the proofs of Proposition 1 and Proposition 2 show that [1], Corollary 1, cannot be extended to Y. Unfortunately, there are many non-uniruled smooth and connected projective surfaces for which $\mathrm{Rat}(Y)$ is infinite (but countable) (e.g. $K3$ surfaces). We were unable to prove that [1], Corollary 1, fails for these surfaces.

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References

[1] J. Bochnak, W. Kucharz, Approximation of holomorphic maps by algebraic morphisms, *Ann. Polon. Math.*, **80** (2003), 85-92.

[2] F.-A. Bogomolov, Families of curves on a surface of general type, *Soviet Math. Dokl.*, **18** (1977), 1294-1297.

[3] O. Debarre, *Higher-Dimensional Algebraic Geometry*, Springer, Berlin-Heidelberg-New York (2001).

[4] J.-P. Demailly, Algebraicity crieteria for Kobayashi hyperbolic projective varieties and jet differentials, In: *Algebraic Geometry Santa Cruz 1995*, 285-360; *Proc. Symp. Pura Math.*, **62**, Part 2, American Mathematical Society, Providence, RI, (1997).

[5] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer, Berlin (1996).