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FIXED POINTS OF MULTIVALUED MAPPINGS IN D-METRIC SPACES

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Abstract: Multivalued mappings in D-metric spaces were introduced by B.C. Dhage [5]. He defined the D-metric version of Hausdorff metric and proved some results. In this paper, we generalize the triangle contraction principle to set-valued mappings in D-metric spaces by defining Hausdorff D-metric in a different way. Consequently, the results so obtained are interesting and different from those of Dhage [5]. We extend the results of H. Kaneko [31], [23] and B.C. Dhage, A.M. Pathan and B.E. Rhoades [8] for multivalued mappings in the setting of D-metric spaces.

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Key Words: D-metric space, fixed point, multivalued mapping, Hausdorff D-metric

1. Introduction

In 1984, B.C. Dhage [3] introduced a notion called D-metric space. The study of this generalized metric space was further enhanced by B.E. Rhoades [25], B.C. Dhage, A.M. Pathan and B.E. Rhoades [8], B.C. Dhage and B.E. Rhoades [9], B.C. Dhage [4], B. Ahmad, M. Ashraf and B.E. Rhoades [2], and B. Ahmad and M. Ashraf [1] by defining some basic properties of D-metric spaces. They

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proved some fixed point theorems for contractive and expansive mappings.

Recall [8] that a nonempty set X, together with a function $D : X^3 \to [0, \infty)$ is called a D-metric space, denoted (X, D), if D satisfies:

(i) $D(x, y, z) = 0$ if and only if $x = y = z$ (coincidence);

(ii) $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry);

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$, for all $x, y, z, a \in X$ (tetrahedral inequality).

The nonnegative real function D is called a D-metric on X.

In 1963, Gahler [10] introduced the concept of a 2-metric space. A function $d : X^3 \to R$ is called a 2-metric [25] on X satisfying (ii) and (iii) above and

(i′) For each distinct pair $x, y \in X$, there exists $z \in X$ such that $d(x, y, z) = 0$, $d(x, y, z) = 0$, if any two of the triplet x, y, z are equal.

It may be noted that geometrically, in plane a 2-metric represents the area of a triangle whereas D-metric represents the perimeter of the triangle with vertices x, y, z. Sufficient literature exists for 2-metric spaces [5], however, C. R. Hsiao [11] pointed out that many fixed point theorems in 2-metric spaces are trivial in the sense that iterations of the mapping f defined therein has the following property:

(H) For all $i, j, k \in I^+$ and for all $x \in X$, $d(fx, fy, fz) = 0$, where $fx = x, f^1x = x, f^2x = f \circ fx$ and so on.

He proved that the theorems mentioned in [12], [15], [19], [20], [17], [18], [24], [28], [27], [29], [30] are trivial because they have the above property (H). The purpose of above note is to emphasize that our present study of D-metric space is different as compared with 2-metric because D-metric and 2-metric are different functions.

2. Some Definitions

Let $f : X \to X$ be a mapping. The orbit of f at the point $x \in X$ is the set $O(x) = \{x, fx, f^2x ...\}$. An orbit of x is said to be bounded if there exists a constant $k > 0$ such that $D(u, v, w) \leq K$ for all $u, v, w \in O(x)$. The constant K is called D-bound of $O(x)$. A D-metric space X is said to be $f-$orbitally bounded if $O(x)$ is bounded for each $x \in X$. A sequence in X is said to be D-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0, such that, for all $m > n$, $p \geq n_0$, $D(x_m, x_n, x) < \varepsilon$. A sequence (x_n) in X is said to be D-convergent to a point $x \in X$ if, for each $\varepsilon > 0$ there exists a positive integer

n_0 such that, for all $m, n \geq 0$, $D(x_m, x_n, x) < \varepsilon$. An orbit $O(x)$ is called f-orbitally complete if every D-Cauchy sequence in converges to a point in X [8]. Let 2^X and $CB(X)$ denote the classes of nonempty closed and bounded subsets of X, respectively. A mapping $T : X \to 2^X$ is called a multivalued mapping on a D-metric space X. A point $u \in X$ is called a fixed point of T if $u \in Tu$, see [6].

In his paper, Dhage [5] showed that some results in D-metric spaces can partially be extended to multivalued mappings. For this, he introduced the concept of Hausdorff D-metric, denoted H_D, as under:

$$H_D(A, B, C) = \max\{\sup_{a\in A, b\in B} D(a, b, C), \sup_{b\in B, c\in C} D(b, c, A), \sup_{c\in C, a\in A} D(c, a, B))\}.$$

Here $D(a, b, C) = \inf\{D(a, b, c) : c \in C\}$. Also

$$D(A, B, C) = \inf\{D(a, b, c) : a \in A, b \in B, c \in C\},$$

and

$$\delta(A, B, C) = \sup\{D(a, b, c) : a \in A, b \in B, c \in C\}.$$

3. Hausdorff D-Metric

If we examine the above definition of Hausdorff D-metric and see its counterpart in metric spaces, it reveals that Dhage fixes one set A and takes sup over the elements of two sets B and C. However, by fixing the two sets and taking the sup over the elements of the third set is more rational and compatible with metric spaces (see definition below).

In view of the above, we have reframed the definitions and applied them successfully to obtain the multivalued version of the triangle contraction principle. In this case, we are able to generalize some basic definitions and ideas for set-valued mappings of metric spaces to D-metric spaces. We define the notions of Hausdorff D-metric, strongly regular orbit, and lower semi continuous mappings in the setting of D-metric spaces.

Let $B(X, D)$ denote the family of all bounded subsets of X. For any sets A, B, C in $B(X, D)$, we define Hausdorff D-metric, denoted H_D, as under:

$$H_D(A, B, C) = \max\{\sup_{x\in A} D(x, B, C), \sup_{x\in B} D(x, C, A), \sup_{x\in C} D(x, A, B)\},$$

where $D(x, B, C) = d(x, A) + d(A, B) + d(x, C)$.

Recall that $d(x, A) = \inf\{d(x, y) : y \in A\}$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

A mapping $T : X \to 2^X$ is continuous in a D-metric space X if for any sequence in X with $x_n \to x$ as $n \to \infty$, we have $\lim_{n\to\infty} H_D(Tx_n, Tx_{n+1}, Tx) = 0$. A function G of a D-metric space X into nonnegative reals is called T-orbitally lower semi-continuous at a point P if (x_n) is a sequence in $O(x)$ and $x_n \to p$ implies $G(p) \leq \text{liminf } G(x_n)$.

4. Fixed Point Theorems

In metric spaces, certain fixed point theorems can partially be extended to multivalued mappings which help to solve many nonlinear problems in game theoretic models in economics, differential and integral calculus and optimization, etc. So, the generalization of point to point mappings to set-valued mappings in D-metric spaces may open a new field of research in many branches of science. Nadler [23] gave the first multivalued version of Banach contraction mapping theorem. Recently, B.C. Dhage and B.E. Rhoades [7] introduced the D-metric version of Banach Contraction Mapping Theorem and named it Triangle Contraction Principle. We will present here the multivalued version of triangle contraction principle. We also generalize the results of H. Kaneko [13], [14] and B.C. Dhage, A.M. Pathan and B.E. Rhoades [4] to set-valued mappings in D-metric spaces. Sufficient literature exists towards generalizing the point to point mappings to multivalued mappings in the study of fixed point theory. S. Nadler [23] defined set valued contraction mappings and proved that for complete metric spaces, these mappings have fixed points. George V. Sehgal and R.E. Smithson [26] extended many results from the single-valued mappings to multivalued mappings by using Liapunov function. Among other contributers in this field are Singh and Whitfield [31], Kaneko [13] and Mizoguchi and Takahashi [21]. S. Nadler [23] generalized the contraction principle to the case of set-valued contraction mappings. Our first theorem is the generalization of triangle contraction principle (Theorem 2, [5]) which is the counterpart of Banach Contraction Theorem in D-metric spaces. Before stating the theorem we define the notion of set-valued Lipschitz mapping in D-metric spaces as follows.

Definition 1. Let (X, D) be a complete D-metric space and H_D the Huasdorff D-metric on the family of all bounded subsets of X. Let $CB(X, D)$ denote the family of all nonempty closed and bounded subsets of X. The set-valued mapping T is called Lipschitz mapping if for all $x, y, z \in X$, there exists a constant k such that

$$H_D(Tx, Ty, Tz) \leq kD(x, y, z).$$

We say that T is a set-valued contraction if $k < 1$.

Following lemma will be required in the sequel.

Lemma 1. (see [4]) *Let (x_n) in X be a bounded sequence with D-bound K satisfying*

$$D(x_n, x_{n+1}, x_m) \leq \lambda^n K,$$

for all positive integers $m > n$ and some $0 \leq x < 1$. Then (x_n) is D- Cauchy.

Now, we are in a position to state our first theorem.

Theorem 1. *Let (X, D) be a complete D-metric space and $CB(X, D)$ the family of all nonempty closed and bounded subsets of X with Hausdorff D-metric H_D. Suppose that $T : X \to CB(X, D)$ is a set-valued Lipschitz mapping. Then T has a fixed point.*

Proof. We select $x_0 \in X$ and $x_1 \in Tx_0$. Then by definition of H_D, there exists $x_2 \in Tx_1$ and $x_m \in Tx_{m-1}$ such that:

$$\begin{aligned} D(x_1, x_2, x_m) &\leq H_D(Tx_0, Tx_1, Tx_{m-1}) + k \\ &\leq kD(x_0, x_1, x_{m-1}) + k, \end{aligned} \tag{1}$$

where $m > n$, for all positive integers m, n. Similarly there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} D(x_2, x_3, x_m) &\leq H_D(Tx_1, Tx_2, Tx_{m-1}) + k \\ &\leq kD(x_1, x_2, x_{m-1}) + k. \end{aligned} \tag{2}$$

Again

$$\begin{aligned} D(x_1, x_2, x_{m-1}) &\leq H_D(Tx_0, Tx_1, Tx_{m-2}) + k \\ &\leq kD(x_0, x_1, x_{m-2},) + k. \end{aligned} \tag{3}$$

Hence from (1), we have

$$\begin{aligned} D(x_2, x_3, x_m) &\leq k\{kD(x_0, x_1, x_{m-2}) + k\} + k \\ &= k^2 D(x_0, x_1, x_{m-2}) + (k^2 + k). \end{aligned} \tag{4}$$

Repeating the argument for all n, we have

$$D(x_n, x_{n+1}, x_m) \leq H_D(Tx_{n-1}, Tx_n, Tx_{m-1}) + k \leq kD(x_{n-1}, x_n, x_{m-1}) + k. \quad (5)$$

Similarly

$$D(x_{n-1}, x_n, x_{m-1}) \leq H_D(Tx_{n-2}, Tx_{n-1}, Tx_{m-2}) + k \leq kD(x_{n-2}, x_{n-1}, x_{m-2}) + k\,.$$

Therefore from (5), we get

$$D(x_n, x_{n+1}, x_m) \leq k\{kD(x_{n-2}, x_{n-1}, x_{m-2}) + k\} + k = k^2 D(x_{n-2}, x_{n-1}, x_{m-2}) + (k^2 + k). \quad (6)$$

Again

$$D(x_{n-2}, x_{n-1}, x_{m-2}) \leq kD(x_{n-3}, x_{n-2}, x_{m-3}) + k. \quad (7)$$

From $(5), (6)$ and (7) we get by induction on m,

$$\begin{aligned} D(x_n, x_{n+1}, x_m) &\leq kD(x_{n-1}, x_n, x_{m-1}) + k \\ &\leq k^2 D(x_{n-2}, x_{n-1}, x_{m-2}) + (k^2 + k) \\ &\leq k^3 D(x_{n-3}, x_{n-2}, x_{m-3}) + (k^3 + k^2 + k) \\ &\leq \cdots \leq k^n D(x_0, x_1, x_{m-n}) + (k^n + ... + k^3 + k^2 + k). \end{aligned}$$

Thus

$$\begin{aligned} D(x_n, x_{n+1}, x_m) &\leq k^n D(x_0, x_1, x_{m-n}) + \{\frac{k(1-k^n)}{1-k}\} \\ &= k^n D(x_0, x_1, x_{m-n}) + k^n\{\frac{k(\frac{1}{k^n} - 1)}{1-k}\} \\ &= k^n K + k^n K' = k^n (K + K')\,, \end{aligned}$$

where $K = D(x_0, x_1, x_{m-n}) > 0$ and $K' = \{\frac{k(\frac{1}{k^n}-1)}{1-k}\} > 0$ are D-bounds of X.

By Lemma 1, it follows that (x_n) is a D-Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$. Since T is continuous, we get

$$x = \lim_{n\to\infty} x_{n+1} \in \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx,$$

which shows that x is the fixed point of T. This completes the proof. □

Theorem 2. *Let $T : X \to P_x(X, D)$ be a mapping of a D-metric space (X, D). Let there exist an $x_0 \in X$ such that $O(x_0)$ is D-bounded and T-orbitally complete. Also suppose that T satisfies*

$$H_D(Tx, Ty, Tz) \leq \lambda \max\{D(x, y, z), D(x, Tx, z)\}, \tag{8}$$

for all $x, y, z \in \overline{O(x_0)}$, for some $0 \leq \lambda < 1$. Then T has a fixed point.

Proof. Let x_0 be arbitrary but fixed point in X. We choose $x_1 \in Tx_0$ and inductively we choose $x_n \in Tx_{n-1}$ for all $m > n$, where m, n are positive integers. Therefore,

$$\begin{aligned} D(x_1, x_2, x_m) &\leq H_D(Tx_0, Tx_1, Tx_{m-1}) \\ &\leq \lambda \times \max\{D(x_0, x_1, x_{m-1}), D(x_0, Tx_0, x_{m-1})\} \\ &= \lambda D(x_0, x_1, x_{m-1}) \leq \lambda K, \end{aligned} \tag{9}$$

where K is D-bound of $O(x_0)$. Again using (8), we have

$$\begin{aligned} D(x_2, x_3, x_m) &\leq H_D(Tx_0, Tx_1, Tx_{m-1}) \leq \lambda \\ &\times \max\{D(x_0, x_1, x_{m-1}), D(x_0, Tx_0, x_{m-1})\} = \lambda D(x_1, x_2, x_{m-1}). \end{aligned} \tag{10}$$

But

$$\begin{aligned} D(x_1, x_2, x_m) &\leq H_D(Tx_0, Tx_1, Tx_{m-2}) \\ &\leq \lambda \max\{D(x_0, x_1, x_{m-2}), D(x_0, Tx_0, x_{m-2})\} \\ &= \lambda D(x_0, x_1, x_{m-2}). \end{aligned} \tag{11}$$

Assuming the induction on m, we have from (8)

$$\begin{aligned} D(x_{n+1}, x_n, x_m) &\leq H_D(Tx_0, Tx_1, Tx_{m-1}) \\ &\leq \lambda \max\{D(x_n, x_{n-1}, x_{m-1}), D(x_n, Tx_n, x_{m-1})\} \\ &= \lambda D(x_n, x_{n-1}, x_{m-1}) \leq \lambda K. \end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned} D(x_n, x_{n+1}, x_{m-1}) &\leq H_D(Tx_{n-1}, Tx_n, Tx_{m-2}) \\ &\leq \lambda \max\{D(x_{n-1}, x_n, x_{m-2}), D(x_{n-1}, Tx_{n-1}, x_{m-2})\} \\ &= \lambda D(x_{n-1}, x_n, x_{m-2}). \end{aligned} \tag{13}$$

Using (12) and (13), we have a recursion formula in m,

$$D(x_{n+1}, x_{n+2}, x_m) \leq \lambda D(x_n, x_{n+1}, x_{m-1}) \leq \lambda^2 D(x_{n-1}, x_n, x_{m-2})$$
$$\leq \cdots \leq \lambda^{n+1} D(x_0, x_1, x_{m-n-1}) = \lambda^{n+1} K,$$

which shows that (x_n) is D-Cauchy. Since X is T-orbitally complete, there exists a point $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Put $x = y = x_n, z = p$ in (8), we get,

$$D(x_{n+1}, x_{n+1}, T_p) \leq H_D(Tx_n, Tx_n, Tx_p)$$
$$\leq \lambda \max\{D(x_n, x_n, p), D(x_n, x_{n+1}, p)\}.$$

Applying the limit as $n \to \infty$, we get $D(p, p, Tp) \leq \lambda D(p, p, p)$ or $D(p, p, Tp) = 0$ which shows that $p \in Tp$. This completes the proof. □

Theorem 3. *Let $T : X \to P_x(X, D)$ be a mapping of a D-metric space (X, D). Let there exist an $x_0 \in X$ such that $O(x_0)$ is D-bounded and T-orbitally complete. Also suppose that for all $x, y, z \in X$,*

$$H_D(Tx, Ty, Tz) \leq aD(x, Tx, z) + bD(x, y, z), \tag{14}$$

where a, b are nonnegative integers and $a + b < 1$ with $a > 1$. Then T has a fixed point.

Proof. Define (x_n) in X as $x_n \in Tx_{n-1}$. Then as proceeded in Theorem 2 above, we see that

$$D(x_{n+1}, x_{n+2}, x_m) \leq qD(x_n, x_{n+1}, x_{m-1}) \leq q^2 D(x_{n-1}, x_n, x_{m-2})$$
$$\leq \cdots \leq q^{n+1} D(x_0, x_1, x_{m-n+1}) = q^{n+1} K,$$

where K is the D-bound of $O(x_0)$. This shows that (x_n) is D-Cauchy sequence in X as X is T-orbitally complete.

Now

$$D(x_{n+1}, Tx, x_{n+1}) \leq H_D(Tx_n, Tx, x_n)$$
$$\leq aD(x_n, x_{n+1}, x_n) + bD(x_n, x, x_n).$$

When $n \to \infty, x_n \to x$, which implies that $D(x, Tx, x) = 0$. Therefore $x \in Tx$. This completes the proof. □

In 1986, H. Kaneko [14] introduced the notion of strongly regular orbit in metric spaces and used it successfully to establish fixed point theorems for many classes of multivalued mappings. In metric spaces, it is not straightforward to show that every strongly regular orbit forms a Cauchy sequence for multivalued mappings. We generalize Kaneko's idea of strongly regular orbit to D-metric spaces in the form of following definition.

Definition. An orbit denoted by $O(x_0)$of a D-metric space is strongly regular if

$$O(x_0) = \{x_n : x_n \in Tx_{n-1;} D(x_n, x_{n+1}, x_m)\} \\ = D(x_n, x_{n+1}, Tx_{m-1})\}, \quad \text{for } (x_n) \text{ in } X, n = 0, 1, 2, \dots .$$

Using the above idea of strongly regular orbit, we generalize the results of H. Kaneko [29] for multivalued mappings in D-metric spaces as Theorem 4 and Theorem 5 below.

Theorem 4. *Let (X, D) be a D-metric space and $T : X \to P_x(X, D)$ a continuous mapping. Suppose that there exists an $x_0 \in X$ such that $O(x_0)$ is D-bounded. If X is T-orbitally complete and α is a monotone increasing function such that $0 \le \alpha < 1$, for all $t \in (0, \infty)$ and if*

$$H_D(Tx, Ty, Tz) \le \alpha(D(x, y, z))D(x, y, z)\,, \tag{15}$$

for each $x, y, z \in X$. Then T has a fixed point.

Proof. Let $x_0 \in X$. Since $T : X \to P_x(X, D)$, we can construct a strongly regular orbit $O(x_0)$ under T. From (14), for any m, we have;

$$D(x_1, x_2, x_m) = D(x_1, x_2, Tx_{m-1}) \le H_D(Tx_0, Tx_1, Tx_{m-1}) \\ \le \alpha(D(x_0, x_1, x_{m+1}))D(x_0, x_1, x_{m-1}).$$

Hence $\{D(x_1, x_2, x_m)\}$ is a monotone decreasing sequence. Similarly

$$D(x_2, x_3, x_m) = D(x_2, x_3, Tx_{m-1}) \le H_D(Tx_1, Tx_2, Tx_{m-1}) \\ \le \alpha(D(x_1, x_2, x_{m-1}))D(x_1, x_2, x_{m-1})\,. \tag{16}$$

But $D(x_1, x_2, x_{m-1}) \le \alpha(D(x_0, x_1, x_{m-2}))D(x_0, x_1, x_{m-2})$. Hence

$$D(x_2, x_3, x_m) \le \alpha(D(x_1, x_2, x_{m-1}))\alpha(D(x_0, x_1, x_{m-2}))D(x_0, x_1, x_{m-2}).$$

This is a recursion formula in n. Thus by induction, for all $m > n$,

$$D(x_{n+1}, x_{n+2}, x_m) = D(x_{n+1}, Tx_{n+1}, Tx_{m-1}) \\ \le H_D(Tx_n, Tx_{n+1}, Tx_{m-1}) \\ \le \alpha(D(x_n, x_{n+1}, x_{m-1}))D(x_n, x_{n+1}, x_{m-1})\,. \tag{17}$$

But

$$D(x_{n+1}, x_{n+2}, x_m) \leq \alpha(D(x_{n-1}, x_n, x_{-m+2}))D(x_{n-1}, x_n, x_{m-2}).$$

Hence (16) becomes

$$D(x_{n+1}, x_{n+2}, x_m) \leq \alpha(D(x_n, x_{n+1}, x_{m-1}))\alpha(D(x_{n-1}, x_n, x_{m-2})) \\ D(x_{n-1}, x_n, x_{m-2})\,. \quad (18)$$

Repeating the argument for all m, we have

$$\begin{aligned} D(x_{n+1}, x_{n+2}, x_m) \leq {} & [\alpha(D(x_n, x_{n+1}, x_{m-1}))\alpha(D(x_{n-1}, x_n, x_{m-2}) \\ & \cdots \alpha(D(x_0, x_1, x_{m-n-1}))]D(x_0, x_1, x_{m-n-1}) \\ & \leq [\alpha(D(x_0, x_1, x_{m-n-1}))]^{n+1} D(x_0, x_1, x_{m-n-1})\,. \quad (19) \end{aligned}$$

Since $\alpha(t) < 1$, we take $\alpha(D(x_0, x_1, x_{m-n-1})) = \lambda < 1$. Hence we have,

$$D(x_{n+1}, x_{n+2}, x_m) \leq \lambda^{n+1} D(x_0, x_1, x_{m-n-1}) = \lambda^{n+1} K. \quad (20)$$

Here (19) is obtained by using the fact that α is monotone increasing function. From (20), it is clear that (x_n) is D-cauchy. Hence (x_n) converges to some element p(say) in X as X is T-orbitally complete. Therefore $\lim_{n\to\infty} x_n = p$.Using the continuity of T, we get

$$p = \lim_{n\to\infty} x_{n+1} \in \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tp.$$

Hence p is the fixed point of T. This completes the proof. □

Theorem 5. *Let $T : X \to P_x(X, D)$ be a mapping of a D-metric space (X, D). If there exists $x_0 \in X$ such that the orbit satisfies*

$$D(x_n, x_{n+1}, x_m) \leq hD(x_{n-1}, x_n, x_{m-1}), \quad (21)$$

for $0 \leq h < 1$ and for each $m > n$. Then

(i) $\lim_{n\to\infty} x_n = p$ exists.

(ii) p is the fixed point of T if and only if $G(x) = D(x, Tx, T^2x)$ is T-orbitally lower semi- continuous at p.

Proof. From (20), it is clear that (x_n) is D-Cauchy . Therefore (x_n) converges to some element p. That is, $\lim_{n\to\infty} x_n = p$. This proves (i). For (ii), since $x_{n+1} \in Tx_n$ and $D(x, B, C) = \{D(x, b, c) : b \in B, c \in C\}$, it follows that $D(x_n, Tx_n, Tx_{n+1}) \leq D(x_n, x_{n+1}, x_{n+2})$. Since $\lim_{n\to\infty} x_n = p$, we have

$$\lim_{n\to\infty} D(x_n, x_{n+1}, x_{n+2}) = 0 \Rightarrow \lim_{n\to\infty} D(x_n, Tx_n, Tx_{n+1}) = 0.$$

Let p be the fixed point of T. Then $G(p) = D(p, Tp, T^2p) = 0 \leq \liminf G(x_n)$, so that G is T-orbitally lower semi-continuous at p. Conversely, let G be T-orbitally lower semi-continuous at p. Therefore

$$G(p) \leq \liminf G(x_n) \leq \lim_{n\to\infty} D(x_n, Tx_n, T^2x_n)$$
$$= \lim_{n\to\infty} D(x_n, Tx_n, Tx_{n+1}) = 0.$$

Hence p must be a fixed point of T. This completes the proof. □

Following theorem classifies D-metric spaces for multivalued mappings.

Theorem 6. *Let $T : X \to P_x(X, D)$ be a mapping such that for all $x, y, z \in X$*

$$H_D(Tx, Ty, Tz) \geq \{D(x, Tx, z)D(y, Ty, z)\}^{\frac{1}{2}},$$

where (X, D) is a D-metric space. Then each $x \in X$ is a fixed point of T.

Proof. We choose $x \in X$. Therefore

$$0 = H_D(Tx, Tx, Tx) \geq \{D(x, Tx, x)D(x, Tx, x)\}^{\frac{1}{2}},$$

or $0 = D(x, Tx, x)$ or $D(x, Tx, x) = 0$, which implies that $x \in Tx$. Hence each $x \in X$ is a fixed point of T. This completes the proof. □

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