SPANNED VECTOR BUNDLES ON REDUCIBLE CURVES

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Abstract: For any reduced curve $C$ and any $P \in C$ let $T_P C$ denote the Zariski tangent space of $C$ at $P$. Let $X$ be a reduced and connected projective curve and $X_i$, $1 \leq i \leq s$, its irreducible components. Here we prove that the following conditions are equivalent:

(i) a vector bundle $E$ on $X$ is spanned if and only if all vector bundles $E|X_i$ are spanned;

(ii) fix an integer $r \geq 1$; a rank $r$ vector bundle $E$ on $X$ is spanned if and only if all vector bundles $E|X_i$ are spanned;

(iii) $h^1(X, \mathcal{O}_X) = \sum_{i=1}^{s} h^1(X_i, \mathcal{O}_{X_i})$;

(iv) the graph of the irreducible components of $X$ is a tree and for any $P \in \text{Sing}(X)$ lying on at least two connected components of $X$, say on $X_{i_j}$ with $1 \leq j \leq x$, we have $\dim(T_P X) = \sum_{j=1}^{x} \dim(T_{P}X_{i_j})$.

AMS Subject Classification: 14J60
Key Words: spanned vector bundle, reducible curve

1. Spanned Bundles on Reducible Curves

It is usually quite difficult to check if a vector bundle on a reducible projective curve is spanned. The aim of this short note is to prove a related result. To
state it we need the following notation. For any reduced curve $C$ and any $P \in C$ let $T_P C$ denote the Zariski tangent space of $C$ at $P$.

**Theorem 1.** Let $X$ be a reduced and connected projective curve and $X_i$, $1 \leq i \leq s$, its irreducible components. The following conditions are equivalent:

(i) a vector bundle $E$ on $X$ is spanned if and only if all vector bundles $E|X_i$, $1 \leq i \leq s$, are spanned;

(ii) fix an integer $r \geq 1$; a rank $r$ vector bundle $E$ on $X$ is spanned if and only if all vector bundles $E|X_i$ are spanned;

(iii) $h^1(X, \mathcal{O}_X) = \sum_{i=1}^s h^1(X_i, \mathcal{O}_{X_i})$;

(iv) the graph of the irreducible components of $X$ is a tree and for any $P \in \text{Sing}(X)$ lying on at least two connected components of $X$, say on $X_i$ with $1 \leq j \leq x$, we have $\dim(T_P X) = \sum_{j=1}^x \dim(T_{P X_i})$.

**Proof.** Obviously (i) implies (ii). For any reduced curve we have $h^1(X, \mathcal{O}_X) \geq \sum_{i=1}^s h^1(X_i, \mathcal{O}_{X_i})$ and $\dim(T_P X) \leq \sum_{j=1}^x \dim(T_{P X_i})$. For any reduced curve $X$ if a vector bundle is spanned, then each vector bundle $E|X_i$ is spanned. It is quite easy to check that (iii) and (iv) are equivalent. Take any line bundle $L$ such that $L|X_i \cong \mathcal{O}_{X_i}$ and any $f \in H^0(X, L)$ if $f|X_i$ is not identically zero, then $f|X_i$ has no zero. If $L$ is spanned, then a general $f \in H^0(X, L)$ vanishes on no irreducible component of $X$ and hence it has no zero. Hence a spanned line bundle $L$ on any reduced curve $X$ is trivial if and only if each $L|X_i$ is trivial. Hence the case $r = 1$ of (ii) implies (iii) (hint: use that $h^1(C, \mathcal{O}_C)$ is the dimension of the tangent space of $\text{Pic}^0(C)$). Now fix an integer $r \geq 2$. Make the same trick starting with $L \oplus \mathcal{O}_X \oplus^{(r-1)}$ and apply the Krull-Schmidt Unique Factorization Theorem ([1], Theorem 3). Hence (ii) for a single $r$ implies (iii). Thus to conclude the proof of Theorem 1 it is sufficient to prove that (iii) implies (i). Since the result is trivial if $X$ is irreducible, we may assume $s \geq 2$ and that the result is true for all connected curves with at most $s - 1$ irreducible components. Fix a vector bundle $E$ on $X$ such that $E|X_i$ is spanned for all $i$. Up to a permutation of the set $\{1, \ldots, s\}$ we may assume that all curves $X^{[m]} := \cup_{i=1}^m X_i$, $1 \leq m \leq s - 1$, are connected. Since $h^1(X, \mathcal{O}_X) \geq h^1(X^{[s-1]}, \mathcal{O}_{X^{[s-1]}}) + h^1(X_s, \mathcal{O}_{X_s})$ and $h^1(X^{[s-1]}, \mathcal{O}_{X^{[s-1]}}) \geq \sum_{i=1}^{s-1} h^1(X_i, \mathcal{O}_{X_i})$, (iii) implies $h^1(X^{[s-1]}, \mathcal{O}_{X^{[s-1]}}) = \sum_{i=1}^{s-1} h^1(X_i, \mathcal{O}_{X_i})$. Hence by the inductive assumption the vector bundle $E|X^{[s-1]}$ is spanned. Since the graph of the irreducible components of $X$ is a tree and $X^{[s-1]}$ is connected, the set $(X^{[s-1]} \cap X_s)_{\text{red}}$ is a unique point, $P$. The second part of (iv) for the
point $P$ implies that $\{P\}$ is the scheme-theoretic intersection of $X^{[s-1]}$ and $X_s$.

Hence we have a Mayer-Vietoris exact sequence on $X$:

$$0 \to E \to E|X^{[s-1]} \oplus E|X_s \to E|\{P\} \to 0. \tag{1}$$

Since $E|X^{[s-1]}$ and $E|X_s$ are spanned, the restriction maps

$$\rho : H^0(X^{[s-1]}, E|X^{[s-1]}) \to H^0(\{P\}, E|\{P\})$$

and

$$\eta : H^0(X_s, E|X_s) \to H^0(\{P\}, E|\{P\})$$

are surjective. Hence by the cohomology exact sequence of the exact sequence (1) we obtain the surjectivity of the restriction maps $\alpha : H^0(X, E) \to H^0(X^{[s-1]}, E|X^{[s-1]})$ and $\beta : H^0(X, E) \to H^0(X_s, E|X_s)$. Since $E|X^{[s-1]}$ is spanned and $\alpha$ is surjective, $H^0(X, E)$ spans $E$ at each point of $X^{[s-1]}$. Since $E|X_s$ is spanned, and $\beta$ is surjective, $H^0(X, E)$ spans $E$ at each point of $X_s$. Hence $E$ is spanned. \qed

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References
