LINEAR INDEPENDENT SUBSCHEMES
FOR VECTOR BUNDLES

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Abstract: There are many possible different definitions of linear independence for subsets of Grassmannians. Grassmannian are associated to spanned vector bundles. Let \( X \) be a projective variety, \( E \) a vector bundle on \( X \), \( V \subseteq H^0(X, E) \) a linear subspace spanning \( E \) and \( Z \subseteq Z' \subset X \) zero-dimensional schemes. Here we explore two notions of “\( Z' \) depends from \( Z \) with respect to \((E, V)\)”, mainly when \( X \) is a smooth curve and \( E \) is stable or semistable.

AMS Subject Classification: 14H60, 14N05
Key Words: vector bundles on curves, Grassmannian, spanned vector bundle, linearly independent zero-dimensional scheme

1. Introduction

Let \( X \) be an integral projective variety, \( E \) a rank \( r \geq 1 \) vector bundle on \( X \) and \( V \subseteq H^0(X, E) \) a linear subspace. Hence \( V \) defines a morphism from \( X \) into an appropriate Grassmannian. There are many possible different definitions of linear independence for subsets of Grassmannians. Here we explore two of them (raising many general questions) in terms of \( X, E, V \), mainly when \( X \) is a smooth curve, \( V = H^0(X, E) \) and \( E \) is stable or semistable. We work over an algebraically closed field \( \mathbb{K} \). For any closed subscheme \( Z \subseteq X \) set \( V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes E) \).
Definition 1. Let $Z \subseteq Z' \subset X$ be zero-dimensional schemes and $Q \in X \setminus Z_{\text{red}}$. We will say that $Z$ is totally independent with respect to the pair $(E, V)$ (or just for $E$ if $V = H^0(X, E)$) if $\dim(V(-Z)) = \dim(V) - r(\text{length}(Z))$. We will say that $Z'$ is totally dependent from $Z$ with respect to $(E, V)$ (or just for $E$ if $V = H^0(X, E)$) if $V(-Z) = V(-Z')$. We will say that $Q$ is partially dependent from $Z$ with respect to the pair $(E, V)$ (or just for $E$ if $V = H^0(X, E)$) if $V(-Z) > V(-Z)$. We will say that $Q$ is partially generically dependent from $Z$ with respect to the pair $(E, V)$ (or just for $E$ if $V = H^0(X, E)$) if $\dim(V(-Z \cup \{Q\})) > \max\{0, \dim(V(-Z)) - r\}$. We will say that $Q$ is partially generically dependent from $Z$ with respect to the pair $(E, V)$ (or just for $E$ if $V = H^0(X, E)$) if $\dim(V(-Z \cup \{Q\})) > \max\{0, \dim(V(-Z \cup \{P\}))\}$ for a general $P \in X$.

The next two examples show that the picture is clear for low genus when we take $V = H^0(X, E)$.

Example 1. Fix integers $r \geq 1$, $a_1 \geq \cdots \geq a_r$, $s \geq 1$, $Z \subset \mathbb{P}^1$ and $Q \in \mathbb{P}^1$ such that $\deg(Z) = s$ and $Q \notin Z_{\text{red}}$. Set $E := \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ and $V := H^0(\mathbb{P}^1, E)$. $Q$ is totally dependent from $Z$ with respect to the pair $(E, V)$ if and only if $s \geq a_1 + 1$. $Q$ is partially dependent from $Z$ with respect to the pair $(E, V)$ if and only if it is totally dependent from $Z$. Notice that these conditions do not depend from the choice of $Z$ and $Q$, but only from the integer $\deg(Z)$ and the assumption $Q \notin Z_{\text{red}}$.

Example 2. Fix integers $r \geq 1$, $d \geq 1$, $s \geq 1$ a smooth elliptic curve $X$, $Z \subset X$, $Q \in X$ and a semistable vector bundle $E$ on $X$ such that $\deg(Z) = s$ and $Q \notin Z_{\text{red}}$, $\text{rank}(E) = r$ and $\deg(E)$. Set $V := H^0(X, E)$. By Atiyah’s classification of vector bundles on any elliptic curve ([1], Part II) we have $h^0(X, E) = d$, $h^0(X, E(-Z)) = \max\{0, d - rs\}$ if $d \neq rs$ and $h^0(X, E(-Z \cup \{Q\})) = \max\{0, d - r(s + 1)\}$ if $d \neq r(s + 1)$. Hence if $d > r(s + 1)$, then $Q$ is not partially dependent from $Z$. Hence if $d \neq r(s + 1)$ the partial or total dependence of $Q$ from $Z$ depends only from the numerical data $r, d, s$ and the condition $Q \notin Z_{\text{red}}$; not from the choice of $Z$, $Q$ and the semistable bundle $E$. Now assume $d = r(s + 1)$ and $E$ indecomposable. We have $h^0(X, E(-Z \cup \{Q\})) \in \{0, 1\}$ and both cases may occur, the second one only for finitely many $Q$‘s (indeed, at most $2^r Q$’s for a fixed $Z$). ([1], Part II). Every vector bundle $F$ on $X$ is the direct sum of indecomposable vector bundles on $X$ ([1]). Hence (as in Example 1) one can easily reduce the problem of partial or total dependence of a pair $(F, H^0(X, F))$ to the knowledge of the ranks and degrees of the indecomposable factors of $F$.

For all integers $g \geq 2$, $r \geq 1$, $d$ and any smooth genus $g$ curve let $M(X; r, d)$ denote the moduli scheme of all stable vector bundles on $X$ with rank $r$ and degree $d$. Hence $M(X; r, d)$ is a non-empty irreducible quasi-projective variety.
of dimension $r^2(g - 1) + 1$.

**Remark 1.** Fix integers $g \geq 2$, $r \geq 1$, $k > 0$, $x \geq 2g + k - 1$ and a smooth genus $g$ curve $X$. For any effective divisor $Z \subset X$ such that $\text{deg}(Z) = k$ and any $L \in \text{Pic}^x(X)$ we have $h^1(X, L(-Z)) = 0$. Hence for all $L_i \in \text{Pic}^x(X)$, $1 \leq i \leq r$, every degree $k$ effective divisor is totally independent for the rank $r$ spanned semistable vector bundle $\oplus_{i=1}^r L_i$.

**Proposition 1.** Assume $\text{char}(\mathbb{K}) = 0$. Fix integers $g \geq 2$, $d \geq 1$, $d - g \geq e > f > 0$, $e - f(d - g - e + f) \geq 0$, $r \geq 1$, and a general smooth genus $g$ curve $X$. Then there are a semistable rank $r$ vector bundle $E$ on $X$ with degree $rd$, $h^1(X, E) = 0$ such that the family of all pairs $(A, B) \in S^{(e-f)}(X) \times S^e(X)$ such that $A \subset B$ and $h^0(X, E(-B)) = h^0(X, E(-A)) = h^0(X, E) - r(e - f)$ (i.e. with $A$ totally independent for $E$ and $B - A$ totally dependent from $A$) is non-empty, reduced and of dimension $e - f(d - g - e + f)$; if $e - f(d - g - e + f) \geq 1$, then it is also irreducible.

**Proof.** For the first part fix a general $L \in \text{Pic}^d(X)$ and set $E := L^{\otimes r}$. Then apply [3], Theorem 0.5, and [4], Theorem B, taking $n := d - g$ and as a $g^n d$ the complete linear system $|L|$. \hfill $\square$

**Remark 2.** Let $X$ be a smooth curve of genus $g \geq 2$, $P \in X$ and $F$ a rank $r$ vector bundle on $X$ such that $h^1(X, F) = 0$. Set $b := \text{deg}(F)$. We have $h^0(X, F(-P)) = h^0(X, F)$ (i.e. $P$ is totally dependent from the empty set with respect to $F$) if and only if $h^1(X, F(-P)) = r$ and this is the case if and only if there is a map $\alpha : F(-P) \to \omega_X^{\otimes r}$ whose image has rank $r$ (i.e. which is injective as a map of sheaves), i.e. if and only if $F(-P)$ is obtained from $\omega_X^{\otimes r}$ applying $r(2g - 2) - b - r$ negative elementary transformations. Obviously, our assumptions imply $r(2g - 2) - b - r \geq 0$. If $r(2g - 2) - b - r = 0$, then we get $F(-P) \cong \omega_X^{\otimes r}$. Now assume $r(2g - 2) - b - r > 0$. The set of all isomorphism classes of vector bundles on $X$ obtained from $\omega_X^{\otimes r}$ making $r(2g - 2) - b - r$ negative elementary transformations is parametrized (perhaps not one-to-one or even not generically finite-to-one) by an irreducible quasi-projective variety of dimension $r(r(2g - 2) - b - r)$. If $\text{char}(\mathbb{K}) = 0$ and $r(2g - 2) - b - r \geq r + 1$, then the general bundle obtained from $\omega_X^{\otimes r}$ making $r(2g - 2) - b - r$ negative elementary transformations is stable (see the proof of Proposition 3).

Inspired from the definitions of Clifford dimension of a line bundle, primitive linear systems and Clifford index of a curve (see [5] and references therein) we pose the following question.
Question 1. Let \( X \) be a smooth and connected genus \( g \geq 2 \) curve. Fix an integer \( r \geq 2 \). Find the minimal integer \( m \geq 2 \) such that there are a rank \( r \) vector bundle \( E \) on \( X \) and effective divisors \( Z \subset Z' \subset X \) such that \( \deg(Z) = n \), \( \deg(Z') \geq 2n - 1 \), \( h^0(X, E) \geq 2rn \), \( Z \) is totally independent for \( E \) and \( Z' \) is totally dependent from \( Z \) with respect to \( E \), i.e. \( h^0(X, E(-Z)) = h^0(X, E) - rn = h^0(X, E(-Z')) \geq rn \). Instead of \( 2rn \) we may take another large integer depending from \( n \) and \( r \). It is more interesting if we find this minimum when we restrict the bundle to a class of bundles satisfying one or more of the following conditions:

1. \( E \) is stable (or semistable);
2. \( E \) is spanned;
3. \( E \) is ample, i.e. the line bundle \( \mathcal{O}_{\mathbb{P}(E^*)}(1) \) on \( \mathbb{P}(E^*) \) is ample;
4. the line bundle \( \mathcal{O}_{\mathbb{P}(E^*)}(1) \) is very ample;
5. \( E \) is spanned and the induced morphism from \( X \) into a Grassmannian is very ample (or birational onto its image).

Instead of zero-dimensional subschemes of \( X \) we may consider the same question in the set-up of the zero-transformations described in Question 2.

Remark 3. Assume \( \text{char}(\mathbb{K}) = 0 \). Fix an integer \( m \geq 3 \). By [5], Theorem 4.3, there are a smooth genus \( 4m - 2 \) curve \( X \) and \( L \in \text{Pic}^{4m-3}(X) \) such that \( L \) is very ample, \( h^0(X, L) = m + 1 \), but there is no \( M \in \text{Pic}^d(X) \) such that \( h^0(X, M) \geq 2 \), \( h^1(X, M) \geq 2 \) and either \( d - h^0(X, M) + 2 \leq 2m - 4 \) or \( d - h^0(X, M) + 1 = 2m - 3 \) and \( h^0(X, M) \leq m \). Taking a linear projection we see that there are no zero-dimensional schemes \( Z \subset Z' \subset X \) such that \( 0 < \deg(Z) \leq m - 2 \), \( \deg(Z') \geq 2(\deg(Z)) \), \( h^0(X, L(-Z)) = h^0(X, L(-Z')) = h^0(X, L) - \deg(Z) \), i.e. \( Z \) is totally independent for \( L \) and \( Z' \) is totally dependent from \( Z \) with respect to \( L \).

Proposition 2. Assume \( \text{char}(\mathbb{K}) = 0 \). Fix integers \( r \geq 2 \) and \( m \geq 3 \). There is a smooth genus \( 4m - 2 \) curve \( X \) and a semistable spanned rank \( r \) vector bundle \( E \) on \( X \) with \( \mathcal{O}_{\mathbb{P}(E^*)}(1) \) very ample, \( \deg(E) = r(4m - 3) \), \( h^0(X, E) = r(m + 1) \) and such that there are no zero-dimensional schemes \( Z \subset Z' \subset X \) such that \( 0 < \deg(Z) \leq m - 2 \), \( \deg(Z') \geq 2\deg(Z) \), \( h^0(X, E(-Z)) = h^0(X, E(-Z')) = h^0(X, E) - r(\deg(Z)) \).

Proof. Take \((X, L)\) as in Remark 3 and set \( E := L^\oplus r \). Apply the last sentence of Remark 3. \(\square\)
Proposition 3. Assume \( \text{char}(\mathbb{K}) = 0 \). Fix integers \( r \geq 2, m \geq 3 \) and \( d \) such that \( r(4m - 3) < d \leq 5rm - 2r + 1, d \neq r(4m - 2) \). There is a smooth genus \( 4m - 2 \) curve \( X \) and a stable rank \( r \) vector bundle \( E \) such that \( \deg(E) = d, h^0(X, E) = r(m + 1) \) and there are no zero-dimensional schemes \( Z \subset Z' \subset X \) such that \( 0 < \deg(Z) \leq m - 2, \deg(Z') \geq 2\deg(Z), h^0(X, E(-Z)) = h^0(X, E(-Z')) = h^0(X, E) - r(\deg(Z)) \).

Proof. Take \((X, L)\) as in Remark 3 and set \( F := L^{\otimes r} \). Let \( E \) be the general bundles obtained from \( F \) making \( d - r(4m - 3) \) positive elementary transformations. By [2], Lemma 2.2, we have \( h^0(X, E) = h^0(X, F) \); here we use \( d \leq r(p_a(X) - 1) + h^0(X, F) \). If \( r(4m - 3) < d < r(4m - 2) \), then \( E \) is stable by [2], Corollary 2.4. The quoted result and [2], Remark 2.5, give the stability of \( E \) for all integers \( d > r(4m - 3) \) such that \( (d - r(4m - 3))/r \notin \mathbb{Z} \). To check the stability of \( E \) in the missing cases by [2], Remark 2.5, it is sufficient to do the case \( d = r(4m - 1) \). Fix \( r \) general \( P_1, \ldots, P_r \in X \) and set \( G := \oplus_{i=1}^r L(P_i) \). Hence \( L(P_i) \not\equiv L(P_j) \) for all \( i \neq j \). \( G \) is obtained from \( F \) making \( r \) positive elementary transformations. Then apply [2], Proposition 2.3, and the openness of stability to obtain the stability of \( E \). Fix a zero-dimensional scheme \( Z \subset X \) such that \( 0 < \deg(Z) \leq m - 2 \) and \( Z \) is totally independent for \( E \). Since \( F \subset E \) and \( h^0(X, F) = h^0(X, E) \), \( Z \) is totally independent for \( E \) and \( Z_{\text{red}} \) does not contain any point over which we make one of the elementary transformations needed to obtain \( E \) from \( F \). To see the non-existence of \( Z' \) as in the statement, use that it does not exists for any factor \( \lambda \) of \( F \), that \( r \geq 2 \) and that the general elementary transformations are supported by different points of \( X \). \( \square \)

Remark 4. Fix an integer \( r \geq 2 \). Let \( X \) be a smooth curve of genus \( g \geq 2 \) and \( L_i \in \text{Pic}(X), 1 \leq i \leq r \). Set \( G := \oplus_{i=1}^r L_i \). \( G \) is not stable and it is semistable if and only if \( \deg(L_i) = \deg(L_j) \) for all \( i \geq j \). Up to a permutation of the set \( \{1, \ldots, r\} \) we may assume \( \deg(L_i) \geq \deg(L_j) \) for all \( i \leq j \). Set \( \Delta := (r - 1)\deg(L_i) - \sum_{j=2}^r \deg(L_j) \) and \( y := \min_{i=1}^r h^1(X, L_i) \). Hence \( \Delta \geq 0 \). We assume \( y > 0 \). For any \( L \in \text{Pic}(X) \) such that \( h^1(X, L) > 0 \) (i.e. such that \( h^0(X, \omega_X \otimes L^*) \not\equiv 0 \)) a general \( P \in X \) is not a base point of \( \omega_X \otimes L^* \). Hence \( h^0(X, L(P)) = h^0(X, L) \) and \( h^1(X, L(P)) = h^1(X, L) - 1 \) for a general \( P \in X \) (Riemann-Roch). Inductively, we get \( h^0(X, L(P_1 + \cdots + P_t)) = h^0(X, L) \) for a general \( (P_1, \ldots, P_t) \in X^{\times t} \) for every integer \( t \) such that \( 1 \leq t \leq h^1(X, L) \). Hence for every integer \( z \) such that \( 1 \leq z \leq y \) we have \( h^0(X, G(P_1 + \cdots + P_z)) = h^0(X, G) \) for a general \( (P_1, \ldots, P_z) \in X^{\times z} \). Fix any such integer \( z \) and any such \( z \)-ple \( (P_1, \ldots, P_z) \in X^{\times z} \). Set \( F := G(P_1 + \cdots + P_z) \). Now assume \( \text{char}(\mathbb{K}) = 0 \) and fix any integer \( x \geq \Delta + r + 1 \). Let \( E \) be the general bundle obtained from \( F \) making \( x \) positive elementary transformations. The bundle \( E \) is stable (see
[2]). We have \( h^1(X, F) = \sum_{i=1}^{r} h^1(X, L_i) - rz \geq r(y - z) \). If \( x \leq h^1(X, F) \), then \( h^0(X, E) = h^0(X, F) \) (see [2], Lemma 2.2). Hence the scheme \( \bigcup_{i=1}^{r} P_i \) is totally dependent for \( E \) if \( x \leq \sum_{i=1}^{r} h^1(X, L_i) - rz \).

We leave to the interested reader the task to obtain results on totally dependent subsets (as in the proofs of Proposition 2 and Proposition 3) from [2] and Remark 4.

**Theorem 1.** Let \( X \) be a smooth curve of genus \( g \geq 2 \). Fix integers \( r \geq 1 \), \( t \geq 1 \) and \( d \geq r(g - 1 + 2t) \). Let \( E \) be the general element of \( M(X; r, d) \). Then every degree \( t \) zero-dimensional \( Z \subset X \) is totally independent for \( E \).

**Proof.** We have \( h^1(X, E) = 0 \). Every vector bundle on the smooth curve \( X \) is the flat limit of a flat family of stable vector bundles on \( X \) (see [6], Corollary 2.2, for a very easy characteristic free proof). Hence it is sufficient to find one rank \( r \) vector bundle \( A \) on \( X \) (even an unstable one) with degree \( d \) and such that \( h^1(X, A(-Z)) = 0 \) for every degree \( t \) zero-dimensional \( Z \subset X \).

(a) Here we assume \( r = 1 \). Fix any \( R \in \text{Pic}^d(X) \) such that \( h^1(X, R) = 0 \) and any degree \( t \) zero-dimensional \( Z \subset X \). \( Z \) is not totally independent for \( R \) if and only if \( h^1(X, R(-Z)) > 0 \), i.e. if and only if there is a divisor \( D \geq 0 \) such that \( L(-Z) \cong \omega_X(-D) \). If \( d - t \geq 2g - 1 \), this is obviously impossible.

If \( d - t \leq 2g - 2 \), then this is equivalent to \( L \cong \omega_X(-B + Z) \) with \( Z > 0 \), \( B \geq 0 \), \( \deg(Z) = t \), \( \deg(B) = 2g - 2 + t - d \). Since \( \dim(\text{Pic}^d(X)) = g \) and \( \dim(S^t(X) \times S^{2g-2+t-d}(X)) = 2g - 2 + 2t - d \leq g - 1 \), there is no such pair \((Z, B)\) for a general \( L \in \text{Pic}^d(X) \).

(b) Here we assume \( r \geq 2 \) and \( d \equiv 0 \) (mod \( Z \)). Take a general \( L \in \text{Pic}^{d/r}(X) \) and set \( A := L^{\oplus r} \). Apply part (a).

(c) Here we assume \( d/r \notin \mathbb{Z} \). By part (b) there is a rank \( r \) vector bundle \( B \) on \( X \) with degree \( r([d/r]) \) and such that \( h^1(X, B(-Z)) = 0 \) for every degree \( t \) zero-dimensional \( Z \subset X \). Take as \( A \) a general bundle obtained from \( B \) making \( d - r([d/r]) \) general positive elementary transformations. \( \square \)

**Definition 2.** Let \( X \) be an integral projective variety, \( E \) a rank \( r \) spanned vector bundle on \( X \) and \( W \subseteq H^0(X, E) \) a linear subspace spanning \( E \). Let \( O_{P(E^*)}(1) \) be the tautological quotient line bundle on \( P(E^*) \). The line bundle \( O_{P(E^*)}(1) \) is spanned and there is a natural isomorphism between \( H^0(X, E) \) and \( H^0(P(E^*), O_{P(E^*)}(1)) \). Hence we may see \( W \) as a linear subspace \( \tilde{W} \) of the vector space \( H^0(P(E^*), O_{P(E^*)}(1)) \) spanning \( O_{P(E^*)}(1) \). We will call zero-transformation of \( E \) any zero-dimensional subscheme of \( P(E^*) \).

**Question 2.** Let \( X \) be an integral projective variety, \( E \) a rank \( r \) spanned vector bundle on \( X \) and \( W \subseteq H^0(X, E) \) a linear subspace spanning \( E \). Set
$m := \dim(W)$. Since the evaluation map $W \otimes \mathcal{O}_X \to E$ is surjective, we have an exact sequence on $X$:

$$0 \to F \to W \otimes \mathcal{O}_X \to E \to 0,$$

with $F$ a rank $(m - r)$ vector bundle on $X$ such that $\det(F) \cong \det(E)^*$. Taking duals in the exact sequence (1) we see that $F^*$ is spanned. What are the relations between dependent zero-transformations for $E$ and dependent zero-transformations for $F^*$?

\section*{Acknowledgements}

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).
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