THE GENERALIZED SEARCH FOR ONE DIMENSIONAL RANDOM WALKER

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Abstract: A target is assumed to move randomly on the real line. Let $X$, the initial position of the target, be a random variable. A searcher starts looking for the target from some point on the line and moves in the two directions from the starting point. Previous studies treated this problem using the origin as the starting point of the search and $X$ had a special distribution. In this paper the problem will be treated in the “general case” which means that the search may start from any point on the real line and for any distribution of $X$, we show the existence of a search plan which minimizes the expected value of the first meeting time of the searcher and the target.

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1. Introduction

Suppose we wish to find lost targets that are either stationary or randomly moving. Problems of this kind have recently applications such as: the search for missing boats, submarines, schools of fish, the search for lost persons on roads, the search for a petroleum or gas reservoirs under ground, and so on (see
Mohamed [12], Ohsumi [14], Washburn [17] and Stone [16]). When the target to be found is stationary or moves randomly on the real line. This problem is of interest because it may arise in many real world situations (see El-Rayes and Abd El-Moneim [8] and Balkhi [2]). Search problem with stationary target on the real line are well studied (see Mohamed and AbuGabl [13], El-Rayes and Abd El-Moneim [8], El-Rayes et al [7], Balkhi [1], [2], Rousseeuw [15], Stone [16], BeckandWarren [4] and Beck and Newman[3]). In the case of randomly moving targets on the real line and the searcher starts the search from the origin, a goal deal of work has been done for deriving conditions for optimal search path which minimizes the cost of effort of finding the target (see El-Rayes and Abd El-Moneim [6], Fristedt and Heath [10] and McCabe [11]). The main contributions of this paper center around studying the search problem for a randomly moving target on the real line where the searcher starts the search from some point on the real line. This problem is defined as generalized search problem for a randomly moving target in one dimension which formulates as follows.

Let \( I \) be a set of integer numbers, and \( I^+ \) be the nonnegative part of \( I \). Assume that \( \{X_i\}_{i \geq 1} \) is a sequence of independent identically distributed random variables such that for any \( i \geq 1 \): \( P(X_i = 1) = p \) and \( P(X_i = -1) = 1 - p = q \), where \( p, q > 0 \). For \( t > 0, t \in I^+ \), \( S(t) = \sum_{i=1}^{t} X_i, S(0) = 0 \) and let \( X \) be a random variable valued in \( 2I \) (or \( 2I + 1 \)) and independent of \( S(t) \), \( t > 0 \). A search plan with speed \( v \) is a function \( \phi : I^+ \to I \) such that:

\[
|\phi(t_1) - \phi(t_2)| \leq v|t_1 - t_2|, \forall t_1, t_2 \in I^+.
\]

The first meeting time \( \tau_\phi \) is a random variable valued in \( I^+ \) which is defined as

\[
\tau_\phi = \begin{cases} 
\inf\{t, \phi(t) = X + S(t)\}, \\
\infty, & \text{if the set is empty.}
\end{cases}
\]

A searcher starts looking for the target from some point \( \phi_0 \). Let \( \Phi_1(t) \) is the set of all search plan with speed equals one. The problem is to find a search plan \( \phi \in \Phi_1(t) \) such that \( E\tau_\phi < \infty \), in this case we call \( \phi \) is a finite search plan and if \( E\tau_\phi^* < E\tau_\phi \forall \phi \in \Phi_1(t) \), where \( E \) terms to the expectation value, then we call \( \phi^* \) is optimal search plan. Given \( n > 0 \), if \( x \) such that:

\[
0 \leq K_1 = \frac{n+x}{2} \leq n,
\]

then

\[
p\{S(n) = K_1\} = \begin{cases} 
\binom{n}{K_1} p^{K_1} q^{n-K_1}, \\
0, & \text{if } K_1 \text{ does not exist.}
\end{cases}
\]
2. Existence of a Finite Search Plan

Let $\lambda$ and $\zeta$ be positive integers such that: $\zeta > 1$, $c = \frac{\zeta - 1}{\zeta + 1} > |p - q|$, $\lambda = k\theta$, where $k = 1, 2, \ldots$, and $\theta$ is the least positive integer such that $\frac{\theta(1 - c)}{2}$ is integer.

We shall define three sequences $\{G_i\}_{i \geq 0}$, $\{r_i\}_{i \geq 0}$, $\{H_i\}_{i \geq 0}$ and a search plan with speed 1 as follows:

$$G_i = \lambda(\zeta^i - 1), \quad r_i = (-1)^{i+1}c[G_i + 1 + (-1)^{i+1}], \quad H_i = r_i + \phi_o, \quad i \geq 1.$$

**Case 1.** We consider $L$ be a set of positive even numbers such that

$$L = \{2, 4, 6, \ldots, n\}, \text{ for } j \in L, \ i \in I^+,$$

we have

$$\cdots < H_{j+2} < 0 < H_j < H_{j-2} < \cdots < H_2 < \phi_o < H_1 < H_3 < \cdots,$$

for any $t \in I^+$ if $G_{2i-1} \leq t < G_{2i}$, $1 \leq i \leq \frac{j}{2}$

$$\phi(t) = H_{2i+1} - \phi_o - [t - G_{2i-1}],$$

if $G_i \leq t < G_{i+1}$, $i \geq j + 1$

$$\phi(t) = H_i + (-1)^i[t - G_i],$$

if $G_{2i} \leq t < G_{2i+1}$, $\leq i \leq \frac{j}{2}$

$$\phi(t) = \phi_o - H_{2i} + [t - G_{2i}].$$

**Case 2.** Let $O$ be a finite set of numbers, such that $O = \{1, 3, 5, \cdots, m\}$, we have, for $j \in O$, $i \in I^+$

$$\cdots < H_2 < \phi_o < H_1 < H_3 < \cdots < H_{j-2} < H_j < 0 < H_{j+2} < \cdots,$$

for any $t \in R^+$, if $G_{2i-1} \leq t < G_{2i}$, $1 \leq i \leq \frac{j+1}{2}$, then

$$\phi(t) = \phi_o - H_{2i-1} - (t - G_{2i-1}),$$

if $G_i \leq t < G_{i+1}$, $i \geq j + 1$, then

$$\phi(t) = H_i + (-1)^i[t - G_i],$$

if $G_{2i} \leq G_{2i+1}$, $1 \leq i \leq \frac{j-1}{2}$, then

$$\phi(t) = H_{2i} - \phi_o + (t - G_{2i}).$$
We use notations, where $K_1(t)$ and $K_2(t)$ are positive functions, $\varphi(t) = S(t) - K_1(t)$ and $\tilde{\varphi}(t) = S(t) + K_2(t)$

**Theorem 2.1.** Let $c$ be a rational number, and let $c$ be different from 1 and $-1$. Define $\forall n \geq 1 V(n) = \{S(n\theta) - cn\theta\}/2$. Then:

(i) there exists a sequence $\{Y_i\}_{i \geq 1}$ of i.i.d.r.v.s such that $V(n) = \sum_{i=1}^{n} Y_i$, and the distribution of $Y_i$ is concentrated on the integers, $E(Y_i) = \frac{\theta(1-c)}{2}(E(X_i) - c)$, and

$$P(Y_i = j > 0 \text{ iff } -\frac{\theta(1+c)}{2} \leq j \leq \frac{\theta(1-c)}{2}.$$

(ii) $p(V(n) = x > 0 \text{ iff } x$ is an integer such that: $\frac{n\theta(1+c)}{2} \leq x \leq \frac{n\theta(1-c)}{2}$.

(iii) If $c \neq p - q$ then there exist constants $r_1$ and $r_2$ depending on $c, p$ such that for any $x \in \mathbb{R}$, if $n > r_1 x + r_2$ then

$$\begin{align*}
p(0 \leq V(n + 1) \leq x) &\leq p(0 \leq V(n) \leq x), \text{ if } x \geq 0, \\
p(x \leq V(n + 1) < 0) &\leq p(x \leq V(n) \leq 0), \text{ if } x \leq 0.
\end{align*}$$

Proof. (i) Define $Y_i = \sum_{j=1}^{\theta}(X_{j+(i-1)\theta} - c)/2, i \geq 1$, $p(Y_i = x) = p(S(\theta) = 2x + \theta c)$. Hence if $P(Y_i = x) > 0$, then by equation (1), $x + \theta(1+c)/2$ is an integer and from the definition of $\theta$, $x$ is an integer also, $0 \leq x + \theta(1+c)/2 \leq \theta$. Therefore $-\theta(1+c)/2 \leq x \leq \theta(1+c)/2$. By using equation (1) the other implication clears. For the last part:

$$E(Y_i) = E\left(\sum_{j=1}^{\theta}(X_{j+(i-1)\theta} - c)/2\right) = \theta(E(X_i) - c)/2.$$

(ii) The second part clears by using equation (1).

(iii) In order to prove the third part of the theorem. If $x \geq 0$, by (ii),

$$\begin{align*}
P(0 \leq V(n) \leq x) &= \sum_{i=0}^{[x]} P(V(n) = i), \\
P(x \leq V(n) < 0) &= \sum_{i=[x]}^{0} p(V(n) = i),
\end{align*}$$

where $[x]$ means the greatest integer less than or equal to $x$. It is sufficient to show that

$$P(V(n + 1) = j) \leq p(V(n) = j) \text{ if } n > r_1 |j| + r_2. \quad (2)$$
We have the following cases:

(a) if $c < -1$, then from (ii), $P(V(n) = j) > 0$ iff $\frac{-2j}{\theta(1+c)} \leq n \leq \frac{-2j}{\theta(1+c)}$ we take $r_1 = \frac{-2}{\theta(1+c)}, r_2 = 0$, then $n > r_1 |j| \Rightarrow n > \frac{-2j}{\theta(1+c)} \Rightarrow P(V(n) = j) = 0$. Consequently (2) holds.

(b) if $c > 1$, hence $P(V(n) = j) > 0$ iff $\frac{-2j}{\theta(1+c)} \leq n \leq \frac{2j}{\theta(1-c)}$, from (ii) put $r_1 = \frac{-2}{\theta(1-c)}, r_2 = 0$, then $n > r_1 |j| \Rightarrow n > \frac{2j}{\theta(1-c)} \Rightarrow p(V(n) = j) = 0$, and (2) holds.

(c) if $-1 < c < 1$, let $\alpha = (1-c)/2, \beta = 1-\alpha$, and $h = (\frac{2}{q})^n (\frac{\beta}{1-q})^\beta$ and put $r_2 = 1/(h-1), r_1 = h \max (1/\alpha, 1/\beta) \theta h - 1)$ since $E(X_i) \neq c$, and $0 < \alpha < 1$, consequently $\alpha \neq q$ and then $h > 1$, so $r_1$ and $r_2$ are positive and well defined. Assume that $n > r_1 |j|$ then $-\beta n \theta \leq j \leq \alpha n \theta$, so $P(V(n) = j) > 0$ from (ii). It remains to prove that: if $P(V(n) = j) > 0$ and $n > r_1 |j| + r_2$, then (2) holds.

Let $Z_i = (X_i - c)/2$ then $P(Z_i = \alpha) = p, P(Z_i = -\beta) = q = 1 - p$ and $P(V(n) = j) = p(\sum_{i=1}^n Z_i = j)$, we can obtain, by using equation (1) and after computations

$$\frac{P(V(n+1) = j)}{P(V(n) = j)} = \prod_{i=1}^{\theta \beta}(\theta n + i)/h\{\theta n + i + (i\alpha + j)/\beta\} \prod_{i=1}^{\theta \beta}(\theta n + i + \beta \theta)/h\{\theta n + i + (i\beta - j)/\alpha\}.$$ 

Since $n > r_1 |j| + r_2$ then every term is strictly less than 1, then

$$p(V(n+1) = j) < p(V(n) = j). \quad \square$$

**Theorem 2.2.** If $E(X_i) < c$, $c \in R$ (the set of real numbers), then there exists $\varepsilon, 0 < \varepsilon < 1$ such that

$$P(S(n) \geq \varepsilon^n \text{ for all } n.$$ 

Proof. For $k > 0$, $P(S(n) \geq cn) = P(\exp\{k(S(n) - nc)\} \geq 1) \leq E \exp\{k(S(n) - nc)\} = \{f(k)\}^n$, where $f(k) = E \exp k\{X_i - c\} = p \exp\{(k(1-c)) + q \exp\{k(-1+c)\}$, if $E(X) < c$, then $f'(0) < 0$, and since $f(0) = 1$, then $\min_{k>0} f(k) = \varepsilon < 1$.

By similar arguments if $E(X_i) > c$, then there exists a positive number $\varepsilon < 1$ such that: $P(S(n) \geq cn) \leq \varepsilon^n$ for all $n$. \quad \square
Theorem 2.3. Let $\gamma$ be the measure defined on $R$ by $X$ and if $\phi(t)$ is the search plan defined above, the expectation $E(\tau_\phi)$ is finite if

$$
\begin{align*}
\int_{-\infty}^{j/2} \sum_{i=1}^{j/2} \zeta^{2i}P(\varphi(G_{2i}) < -x) & + \sum_{i=j/2+1}^{\infty} \zeta^{2i}P(\tilde{\varphi}(G_{2i} \leq -x))\gamma(dx), \\
\int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \zeta^{2i+1}P(\varphi(G_{2i+1}) > -x)\gamma(dx), & \quad \text{where } \phi > 0,
\end{align*}
$$

are finite.

Proof. The hypothesis $X$ and $\phi_o$ are valued in $2I$ or $(2I + 1)$ then $X + S(t)$ is greater than $\phi(t)$ until the first meeting, also if $X$ is smaller than $\phi_o$ then $X + S(t)$ is smaller than $\phi(t)$ until the first meeting. Hence for any $i \geq 0$

$$
P(\tau_\phi > G_{2i+1}) \leq \int_{-\infty}^{\phi_o} P\{X + S(G_{2i}) < H_{2i}/X = x\} \gamma(dx)$$

$$
+ \int_{\phi_o}^{\infty} P\{X + S(G_{2i+1}) > H_{2i+1}/X = x\} \gamma(dx).
$$

Then we can get

$$
E(\tau_\phi) = \int_{0}^{\infty} P(\tau_\phi > t)dt \leq \sum_{i=0}^{G_{i+1}} \int_{G_{i}}^{G_{i+1}} P(\tau_\phi > t)dt \leq \sum_{i=0}^{G_{i+1}} \int_{G_{i}}^{G_{i+1}} P(\tau_\phi > G_{i})dt
$$
$$\sum_{i=0}^{\infty} (G_{i+1} - G_i) P(\tau_\phi > G_i) = \lambda \sum_{i=0}^{\infty} (\zeta^{i+1} - \zeta^i) P(\tau_\phi > G_i)$$

$$= \lambda (\zeta - 1) \sum_{i=0}^{\infty} \zeta^i P(\tau_\phi > G_i) = \lambda (\zeta - 1)[P(\tau_\phi > 0) + \zeta P(\tau_\phi > G_1)$$

$$\quad + \zeta^2 P(\tau_\phi > G_2) + \zeta^3 P(\tau_\phi > G_3) + \cdots].$$

If $\phi_\circ < 0$, then

$$E(\tau_\phi) \leq \lambda (\zeta - 1)[P(\tau_\phi > 0) + \zeta P(\tau_\phi > G_1)$$

$$\quad + \zeta^2 \{ \int_{\phi_\circ}^{\infty} p(\phi(G_1) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_2) < -x) \gamma(dx) \}$$

$$\quad + \zeta^3 \{ \int_{\phi_\circ}^{\infty} p(\phi(G_3) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_2) < -x) \gamma(dx) \}$$

$$\quad + \zeta^4 \{ \int_{\phi_\circ}^{\infty} p(\phi(G_3) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_4) < -x) \gamma(dx) \}$$

$$\quad + \cdots + \zeta^j \{ \int_{\phi_\circ}^{\infty} p(\phi(G_j) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_{j-1}) < -x) \gamma(dx) \}$$

$$\quad + \zeta^{j+1} \{ \int_{\phi_\circ}^{\infty} p(\phi(G_{j+1}) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_j) < -x) \gamma(dx) \}$$

$$\quad + \zeta^{j+2} \{ \int_{\phi_\circ}^{\infty} p(\phi(G_{j+2}) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_{j+1}) < -x) \gamma(dx) \}$$

$$\quad + \zeta^{j+3} \{ \int_{\phi_\circ}^{\infty} p(\phi(G_{j+3}) > -x) \gamma(dx) + \int_{-\infty}^{\phi_\circ} p(\phi(G_{j+2}) < -x) \gamma(dx) \}$$

$$\quad + \cdots E(\tau_\phi) \leq \lambda (\zeta - 1)[\bar{g} + (\zeta + 1)\zeta^2 \int_{-\infty}^{\phi_\circ} p(\phi(G_2) < -x) \gamma(dx)$$
\[ E(\tau_\phi) \leq \lambda (\zeta - 1) \left[ \tilde{g}(\zeta + 1) \left\{ \int_{-\infty}^{\phi_o} m(x) \gamma(dx) + \int_{\phi_o}^{\infty} q(x) \gamma(dx) + \int_{\phi_o}^{\infty} \tilde{q}(x) \gamma(dx) \right\} \right], \]

where

\[ \tilde{g} = p(\tau_\phi > 0) + \zeta p(\tau_\phi > G_1) + \zeta^2 \int_{\phi_o}^{\infty} p(\tilde{\varphi}(G_1) > -x) \gamma(dx), \]
\[ m(x) = \zeta^2 p(\tilde{\varphi}(G_{2i}) < -x), \]
\[ q(x) = \sum_{i=1}^{(j-1)/2} \zeta^{2i+1} P[\tilde{\varphi}(G_{2i+1}) > -x], \]
\[ \tilde{q}(x) = \sum_{i=(j+1)/2}^{\infty} \zeta^{2i+1} P[\varphi(G_{2i+1}) > -x]. \]

By similar way if \( \phi_o > 0 \), then
\[ E(\tau_\phi) \leq \lambda(\zeta - 1)[g + (\zeta + 1)\{\int_{-\infty}^{\phi_0} v(x) \gamma(dx) + \int_{-\infty}^{\phi_0} \tilde{v}(x) \gamma(dx) + \int_{\phi_0}^{\infty} w(x) \gamma(dx)\}], \]

where
\[ v(x) = \sum_{i=1}^{j/2} \zeta^{2i} p[\varphi(G_{2i}) > -x], \]
\[ \tilde{v}(x) = \sum_{i=1}^{\infty} \zeta^{2i} p[\tilde{\varphi}(G_{2i}) > -x], \]
\[ w(x) = \sum_{i=1}^{\infty} \zeta^{2i+1} p[\varphi(G_{2i+1}) > -x], \]
\[ g = p(\tau_\phi > 0) + \zeta p(\tau_\phi > G_1) + \zeta^2 \int_{\phi_0}^{\infty} p(\varphi(G_1) > -x) \gamma(dx). \]

**Lemma 1.** (see El-Rayes et al [5]) Let \( a_n \geq 0 \) for \( n \geq 0 \), and \( a_{n+1} \leq a_n \), \( \{d_n\}_{n \geq 0} \) be a strictly increasing sequence of integers with \( d_0 = 0 \), then for any \( k \geq 0 \),
\[ \sum_{n=k}^{\infty} [d_{n+1} - d_n] a_{d_{n+1}} \leq \sum_{n=d_k}^{\infty} a_n \leq \sum_{n=k}^{\infty} [d_{n+1} - d_n] a_{d_n}. \]

**Theorem 2.4.** The chosen search plan satisfies
\[ w(x) \leq L(|x|), \quad \tilde{v}(x) \leq L'(|x|), \quad \phi_0 > 0 \]
\[ m(x) \leq L''(|x|) \text{ and } \tilde{q}(x) \leq L'''(|x|), \quad \phi_0 < 0, \]
where \( L(|x|), L'(|x|), L''(|x|) \) and \( L'''(|x|) \) are linear functions.

**Proof.** We shall prove the theorem for \( w(x) \)
\[ w(x) = \sum_{i=1}^{\infty} \zeta^{2i+1} p[\varphi(G_{2i+1}) > -x], \quad \phi_0 > 0. \]
(i) If $x \geq \phi_0$:

$$w(x) = w(\phi_0) + \sum_{i=1}^{\infty} \zeta^{2i+1} p[-x < \varphi(G_{2i+1}) \leq -\phi_0].$$

(ii) If $0 \leq x \leq \phi_0$:

$$w(x) = w(0) + \sum_{i=1}^{\infty} \zeta^{2i+1} p[-x < \varphi(G_{2i+1}) \leq 0].$$

(iii) If $x \leq 0$:

$$w(x) = w(0) - \sum_{i=1}^{\infty} \zeta^{2i+1} p[0 < \varphi(G_{2i+1}) \leq -x].$$

We have for $x \geq \phi_0$, where in (ii) $w(x) \leq w(0)$,

$$w(x) = w(\phi_0) + \sum_{i=1}^{\infty} \zeta^{2i+1} p[-x \leq -\phi_0],$$

but, $w(x) = w(0) + \sum_{i=1}^{\infty} \zeta^{2i+1} p[-x < \varphi(G_{2i+1}) \leq 0]$.

From Theorem 2.2, we get

$$w(0) = \sum_{i=1}^{\infty} \zeta^{2i+1} p[\varphi(G_{2i+1}) > 0] \leq \sum_{i=1}^{\infty} \zeta^{2i+1} \epsilon G_{2i+1} \leq \frac{\zeta^3}{\zeta^2 - 1},$$

$0 < \epsilon < 1$.

We define the following:

(1) $d_n = G_{2n+1}/\theta = k(\zeta^{2n+1} - 1);$  
(2) $k(n) = \varphi(n\theta)/2 = \sum_{i=1}^{n} Y_i$, where $\{Y_i\}_{i \geq 1}$ is a sequence of (i.i.d.r.v.);  
(3) $a(n) = p[-x/2 < k(n) \leq 0] = \sum_{j=0}^{\lfloor x/2 \rfloor} p[-(j+1) < k(n) \leq -j];$  
(4) $m$ is an integer such that $d_m = r_1 |x| + r_2;$  
(5) $\cup(j, j+1) = \sum_{n=0}^{\infty} t P[-(j+1) < k(n) \leq -j];$  
(6) $\alpha = \zeta^2/(\zeta^2 - 1)k.$
If \( n > d_n \) then by Theorem 2.1, \( a(n) \) is nonincreasing, and we can apply Lemma 1, in the suitable step,

\[
w(x) - w(0) = \sum_{i=1}^{\infty} \zeta^{2i+1} p[-x < \varphi(G_{2i+1}) \leq -a] = \\
\sum_{n=1}^{m} \zeta^{2n+1} a(d_n) + \sum_{n=m+1}^{\infty} \zeta^{2n+1} a(d_n) \leq \sum_{n=1}^{m} \zeta^{2n+1} + \alpha \sum_{n=m+1}^{\infty} (d_n - d_{n-1}) a(d_n) \\
\leq \sum_{n=1}^{m} \zeta^{2n+1} + \alpha \sum_{n=d_m}^{\infty} a(n) \leq \sum_{n=1}^{m} \zeta^{2n+1} + \alpha \sum_{j=0}^{\infty} U(j, j + 1).
\]

Since \( U(j, j + 1) \) satisfies the conditions of renewal theorem (see Feller [9]), hence \( U(j, j + 1) \) is bounded for all \( j \), by a constant, so

\[
w(x) \leq w(\varphi_{\phi}) + M_1 + M_2 |x| = L(|x|). \quad \Box
\]

**Theorem 2.5.** If there exist a finite search plan then \( E|X| \leq \infty \).

*Proof.* If \( E\tau_{\phi} < \infty \), then \( P(\tau_{\phi} < \infty) = 1 \), and so \( X = \phi(\tau_{\phi}) - S(\tau_{\phi}) \), with probability 1, so

\[
|X| \leq |\phi(\tau_{\phi})| + |S(\tau_{\phi})| \leq \tau_{\phi} + |S(\tau_{\phi})|.
\]

Hence \( E|X| \leq E\tau_{\phi} + E|S(\tau_{\phi})| \), but \( |S(\tau_{\phi})| \leq \tau_{\phi} \), then \( E|S(\tau_{\phi})| \leq E\tau_{\phi} \), and \( E|X| < \infty \). \( \Box \)

**Remark.** A direct consequence of Theorems 2.3, 2.4, and 2.5 is such that there exist a finite search plan iff \( E|X| < \infty \).

### 3. Existence of an Optimal Search Path

**Definition.** Let \( \phi_n \in \Phi(t) \) be a sequence of search plans, we say that \( \phi_n \) converges to \( \phi \) as \( n \) tends to \( \infty \) iff for any \( t \in I^+ \), \( \phi_n(t) \) converges to \( \phi(t) \) uniformly on every compact subset.

**Theorem 3.1.** Let for any \( t \in I^+ \), \( S(t) \) be a process. The mapping

\[
\phi \longrightarrow E(\tau_{\phi}) \in R^+
\]

is lower semicontinuous on \( \phi(t) \).
Proof. Let $I(\phi, t)$ be the indicator function of the set $\{\tau_\phi \geq t\}$, by the Fatou-Lebesque Theorem we get

$$E(\tau_\phi) = E[\sum_{t=1}^{\infty} I(\phi, t)] = E[\sum_{t=1}^{\infty} \lim_{n \to \infty} \inf I(\phi_n, t)] \leq \lim_{n \to \infty} \inf E(\tau_{\phi_n})$$

for any sequence $\phi_n \to \phi$ in $\Phi_1(t)$, where $\Phi_1(t)$ is sequentially compact (see El-Rayes et al [5]), thus the mapping $\phi \to E(\tau_\phi)$ is lower semicontinuous mapping on $\Phi_1(t)$, then this mapping attains its minimum. \qed

References


