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ON THE UBIQUITY OF UNSTABLE LOCALLY FREE COHERENT SYSTEMS ON INTEGRAL PROJECTIVE VARIETIES

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Abstract: Fix integers $k \geq 1$, $n \geq 2$, an integral projective variety X, a rank n torsion free sheaf E on X and an ample line bundle H on X. Here we prove the existence of an integer t_0 and a positive real number α_0 (depending only from X, E and H) such that for all integers $t \geq t_0$ there is a k-dimensional linear subspace $W \subseteq H^0(X, E(tH))$ such that the cogent pair $(E(tH), W)$ is not α-stable (with respect to the polarization H) for all real numbers $\alpha \geq \alpha_0$.

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1. Unstable Coherent Systems

For the general theory of coherent systems, see [1], [2] and [3]. Here we work over an algebraically closed field $\mathbb{K}$ and prove the following result.

Theorem 1. *Fix integers $k \geq 1$, $n \geq 2$, an integral projective variety X, a rank n torsion free sheaf E on X and an ample line bundle H on X. Then there is an integer t_0 and a positive real number α_0 (depending only from k, X, E and H) such that for all integers $t \geq t_0$ there is a k-dimensional linear subspace $W \subseteq H^0(X, E(tH))$ such that the cogent pair $(E(tH), W)$ is not α-stable (with respect to the polarization H) for all real numbers $\alpha \geq \alpha_0$.*

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Proof of theorem 1. Set $x := \dim(X)$. For any torsion free sheaf $F \neq 0$ on X we the rank one torsion free sheaf $\det(F)$ may be used to define the slope $(\det(F) \cdot H^{x-1})/\text{rank}(F)$ and define in this way a notion of stability. We will use this notion, although the notion obtained using the Hilbert polynomial of F is better from the point of view of moduli spaces. Theorem 1 is true for both notions of stability with respect to the polarization H, but the proof for the slope stability is simpler. Fix a linear subspace $W \subseteq H^0(X, F)$ and a real number α. Set $\mu_\alpha(F, W) = \mu(F) + \alpha(\dim(W)/\text{rank}(F))$ and use this notation to define the α-stability of a coherent system with respect to the polarization H. Since X is projective, there is a rank one subsheaf F of E. Set $\beta := \mu(E) - \mu(F)$ and take as α_0 any real number such that $\alpha_0 \cdot k(n-1)/n > \beta$. It is sufficient to have $\alpha_0 \cdot (n-1)/n > \beta$ and hence α_0 does not depend from k. We have $\mu(E(tH)) - \mu(F(tH)) = \beta$ for all t. Since H is ample, there is an integer t_0 such that $h^0(X, F(tH)) \geq n$. Fix any such integer t and any n-dimensional linear subspace W of $H^0(X, F(tH)) \subseteq H^0(X, E(tH))$. By construction the pair $(F(tH), W)$ shows that the pair $(E(tH), W)$ is not semistable. □

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